# MATHEMATICAL

# QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

## PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

*PUBLISHED IN THE "EDUCATIONAL TIMES,"*

WITH AN

## APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

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# CONTENTS.

## Mathematical Papers, &c.



## Solved Questions.

## APPENDIX.



*CORRIGENDUM.*

Page 111, line 24, *for* 10032 *read* 10320.

# MATHEMATICS

FROM

## THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

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**8184.** (Professor SYLVESTER, F.R.S.)—Prove that (1) the equations between two corresponding points, in two systems homologically related, may be put under the form

$$x' = \rho x + f(ax+by+cz),\quad y' = \rho y + g(ax+by+cz),\quad z' = \rho z + h(ax+by+cz);$$

and (2) if two homographic systems of points connected by the equations $x' = Ax$, $y' = By$, $z' = Cz$, are moved into homology, the condition that the pole shall be contained in the axis of homology is expressed by

$$[AB(A-C)(B-C)p^2 + BC(B-A)(C-A)q^2 + CA(C-B)(A-B)r^2]^3$$
$$+ A^2B^2C^2(A-C)(B-C)p^2 + (B-A)(C-A)q^2 + (C-B)(A-B)r^2]^3 = 0,$$

where $p$, $q$, $r$ are the lengths of the sides of the fundamental triangle.

Mr

---

*Solution by* Professor SEBASTIAN SIRCOM.

(1) Evidently the axis is $ax+by+cz$, the pole $f, g, h$. (2) The absolute values of the corresponding coordinates will be $2\Delta Ax/(Apx+Bqy+Crz)$, &c.

The circular points at infinity $\omega$, $\omega'$ being given by $px+qy+rz = 0$ and $x^2+y^2+2xy\cos\gamma = 0$, the corresponding points $o$, $o'$ by $Apx+Bqy+Crz = 0$, $A^2x^2+B^2y^2+2ABxy\cos\gamma = 0$, the lines $\omega$, $o$; $\omega'$, $o'$ intersect in the point

$$\frac{A(B-C)x_1}{p} = \frac{2\Delta ABC(B-A)(C-A)(B-C)}{M}, \text{ \&c.,}$$

and the corresponding point will be

$$\frac{(B-C)x_2}{p} = \frac{2\Delta(B-A)(C-A)(B-C)}{N}, \text{ \&c.,}$$

where

$$M = BC(B-A)(C-A)p^2 + CA(C-B)(A-B)q^2 + AB(A-C)(B-C)r^2,$$
$$N = (B-A)(C-A)p^2 + (C-B)(A-B)q^2 + (A-C)(B-C)r^2.$$

Since the circular points at infinity are unmoved by any change of position of the fundamental triangle, let the triangle be moved parallel to itself so that the vertex C coincides with $x, y, z$; then, leaving $x, y$ unaltered, increase the corresponding coordinates $x', y'$ by $x_1-x_2, y_1-y_2$; then (SALMON, *Higher Plane Curves*, p. 300) C will be the centre of homology, and the absolute values of the coordinates corresponding to $x, y$, will be

$$x' = \frac{2\Delta(x-x_1)}{Ap(x-x_1)+Bq(y-y_1)+C[2\Delta-p(x-x_1)-q(y-y_1)]}-x_2,\quad y'=\&c.$$

Using the values of $x_1, y_1, x_2, y_2$, these become

$$x' = \frac{[2\Delta A-(A-C)px_2]\,x-(B-C)\,qx_2y}{(A-C)\,px+(B-C)\,qy-2\Delta ABCN/M},$$

$$y' = \frac{[2\Delta B-(B-C)\,qy_2]\,y-(A-C)\,py_2x}{(A-C)\,px+(B-C)\,qy-2\Delta ABCN/M}.$$

The systems will now be brought into homology by turning the second about C through an angle $\theta$ determined by

$$\frac{2\Delta A-(A-C)\,px_2}{\sin(\gamma-\theta)}=\frac{(B-C)\,qx_2}{\sin\theta}=\frac{2\Delta B-(B-C)\,qy_2}{\sin(\gamma+\theta)}=-\frac{(A-C)\,py_2}{\sin\theta};$$

each of these ratios will be found to be equal to $-(M/N)^{\frac{1}{2}}pq$; then, since $pq\sin C=2\Delta$, the transformed coordinates will be

$$x'=2\Delta M^{\frac{3}{2}}x/(X+ABCN^{\frac{3}{2}}rz),\quad y'=2\Delta M^{\frac{3}{2}}y/(X+ABCN^{\frac{3}{2}}rz),$$

where X does not contain $z$; then

$$rz'=2\Delta-px'-qy'=2\Delta\,[1-M^{\frac{3}{2}}(px+qy)/(X+ABCN^{\frac{3}{2}}rz)]$$

$$=\frac{2\Delta M^{\frac{3}{2}}}{X+ABCN^{\frac{3}{2}}rz}\left[rz+\frac{rz(ABCN^{\frac{3}{2}}-M^{\frac{3}{2}})}{M^{\frac{3}{2}}}+X_1\right],$$

then, by (1), $X_1+rz(ABCN^{\frac{3}{2}}-M^{\frac{3}{2}})/M^{\frac{3}{2}}=0$ is the axis of homology which passes through $x=0,\ y=0$, if $A^2B^2C^2N^3=M^3$; or

$$\{BC(B-A)(C-A)\,p^2+CA(C-B)(A-B)\,q^2+AB(A-C)(B-C)\,r^2\}^3$$
$$=A^2B^2C^2\{(B-A)(C-A)\,p^2+(C-B)(A-B)\,q^2+(A-C)(B-C)\,r^2\}^3,$$

which appears to be right.

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NOTE ON QUESTION 9588. (Vol. XLIX., pp. 101-2).

*By the* EDITOR.

We give here in regard to this Question, a *final* rejoinder and counter-rejoinder, as our space, we regret to say, will not allow a word more to either of the antagonists.

1. Professor TANNER states that "in Mr. DODGSON's note he considers two aggregates of points; viz., (I.) the points dividing a given line into two commensurable parts, and (II.) the points that divide the line into two incommensurable parts; and he asserts that a random point in the line must coincide with one of the points of (I.) or of (II.) I take this to mean that every point of the line belongs either to (I.) or to (II.); that is

to say, the two aggregates make up the whole line. This is inconsistent with Mr. Dodgson's axiom 2, and I submit that, in his attempt to convict the 'opposition' of contradicting one axiom, he has himself contradicted the other.

In the second part of my note (Vol. L., p. 34) I assumed that a random point could not be in two places at once, and therefore applied the common theory of the probabilities of mutually exclusive events. I am not clear, even after studying the 'fair instance,' whether it is the assumption or the theory to which exception is taken in Mr. Dodgson's reply (Vol. L., p. 35)."

2. Mr. Dodgson replies thus:—"I agree with this Note as far as the words, 'every point of the line belongs either to (I.) or to (II.)'; but I deny the next clause, viz., 'that is to say, the two aggregates make up the whole line.' This does *not* follow, unless we assume as an axiom that the aggregate of the points in a line make up the line; and this I entirely deny. I hold that a line is the aggregate, not of the *points* in it, but of the *distances between those points.*

In the second paragraph, I take no 'exception' to the assumption 'that a random point cannot be in two places at once,' nor yet to the 'theory of the probabilities of mutually exclusive events;' but I hold that the Professor is mistaken (see Vol. L., p. 34) in regarding $n$ as greater than unity. If $\delta$ represent 'the chance of a random point in a given line of unit length coinciding with an assigned point,' the number of its possible positions must be $1/\delta$, and each must be distant $\delta$ from its neighbour; *i.e*, $\delta/n = \delta$, and $n = 1$. Hence, what he calls 'a segment of length $1/n$' is really *the whole line*, and there is no 'absurdity' to 'avoid.'

It would be very interesting to know which (if any) of the following three statements the Professor accepts.

For brevity, let 'an $\alpha$-point' mean one that divides a given line into two commensurable parts, and 'a $\beta$-point' one that divides it otherwise.

(1) No aggregate, however numerous, of *absolute zeroes* can constitute a magnitude.

(2) The chance of a random point on a line coinciding with an assigned point is *absolute zero.*

(3) The chance of a random point on a line being of one of the two kinds, either an $\alpha$- or else a $\beta$-point, is unity.

If he accepts all three, I beg to offer him the following pit-fall:—

(4) [logically deducible from (1), (2).] The chance of a random point on a line being one of the aggregate of $\alpha$-points is *absolute zero.*

(5) [ditto.] The chance of its being one of the aggregate of $\beta$-points is *absolute zero.*

(6) [logically deducible from (4), (5).] The chance of its being of one of the two kinds, either an $\alpha$- or else a $\beta$-point, is *absolute zero.*

(7) [logically deducible from (3), (6).] Unity is *absolute zero.*"

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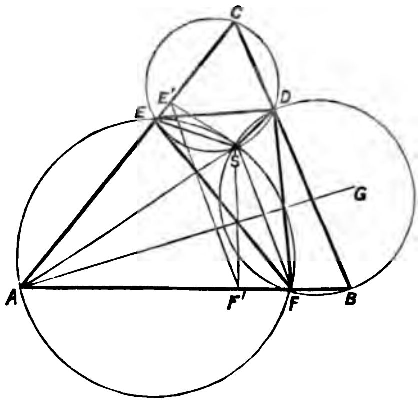
**10056.** (Prof. Syamadas Mukhopâdhyây, B.A.)—A system of triangles of given species are inscribed in a given triangle; prove that (1) the envelope of each side is a parabola which touches the two sides of the given triangle including that side; (2) the three parabolas have a

common focus which is the common centre of homology of the system of inscribed triangles; (3) their vertical tangents form the minimum triangle of the system; and (4) that diameters of these parabolas through corresponding vertices of the given triangle meet at a point which is isogonally conjugate to the common centre of homology.

*Solution by* Professor SCHOUTE.




If the triangle DEF inscribed in triangle ABC is given in species, the point S common to the circles AEF, BFD, CDE is fixed. For this point, be it fixed or not, is a point of the same signification for all the triangles DEF of the series, the point common to the segments of the circles described on EF, FD, DE, and capable of the angles supplementary to A, B, C. But, as $\angle$ CAS = $\angle$ EFS, $\angle$ SFD = $\angle$ SBC, S is fixed. So the ranges of points E, E'... on AC, and F, F'... on BA, can be generated by the rotation of an invariable angle ESF, which proves that EF envelopes a parabola of which S is the focus and AC and BA are tangents. This common focus of the parabolas generated by EF, FD, DE is the common centre of homology of any couple of triangles DEF. In the position E'SF' of the rotating angle, the lines SE' and SF' pass through their minimum value; therefore the tangent E'F' is the tangent at the vertex of the parabola, this tangent being the tangent of minimum distance to the focus, which proves that the tangents at the vertices of the three parabolas include the inscribed triangle of minimum surface. And, finally, the perpendicular AG on E'F' is isogonal conjugate to AD with reference to angle A, etc.

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**10054.** (W. S. M'CAY, M.A.)—Soient AB un diamètre d'une circonférence, et CD une corde perpendiculaire à AB au point E; si $a, b, c, d$ sont les distances des points A, B, C, D à une droite quelconque L et ($\lambda : \mu$) le rapport des segments BE, AE, (1) démontrer que

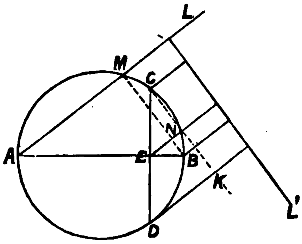
$$\lambda a^2 + \mu b^2 = (\lambda + \mu)\left[cd + (\mathrm{CE})^2\right];$$

(2) déduire de là l'enveloppe d'une droite L satisfaisant à la condition

$$\lambda a^2 + \mu b^2 = \text{const.}$$

*Solution by* L. WIENER; G. G. STORR, M.A.; *and others.*

Draw CK perpendicular to $d$, and join B to M, where $a$ meets the circle;




then $\qquad \dfrac{\mathrm{CK}}{2\mathrm{CE}} = \dfrac{am}{\mathrm{AB}} = \dfrac{a-b}{\mathrm{AB}},$

therefore $(\mathrm{CK})^2 = 4(\mathrm{CE})^2 \dfrac{(a-b)^2}{(\mathrm{AB})^2}$;

but $\quad (\mathrm{CE})^2 = \mathrm{AE}.\mathrm{EB}$

$$= \frac{\mu}{\lambda+\mu}\mathrm{AB} \times \frac{\lambda}{\lambda+\mu}\mathrm{AB},$$

hence $(\mathrm{CK})^2 = \dfrac{4\lambda\mu}{(\lambda+\mu)^2}(a-b)^2.$

But $\qquad (\mathrm{CD})^2 = 4(\mathrm{CE})^2 = (\mathrm{CK})^2 + (\mathrm{DK})^2 = \dfrac{4\lambda\mu}{\lambda+\mu}(a-b)^2 + (d-c)^2,$

hence $$\frac{\lambda}{\lambda+\mu}a^2 + \frac{\mu}{\lambda+\mu}b^2 = cd + \frac{\lambda\mu}{\lambda+\mu}(a-b)^2 + \left(\frac{d-c}{2}\right)^2 \quad \text{............}(\alpha).$$

$$\frac{\mathrm{EB}}{\mathrm{AB}} = \frac{\lambda}{\lambda+\mu} = \frac{\mathrm{EN}}{am} = \frac{\frac{1}{2}(c+d)-b}{a-b}. \qquad \text{Similarly} \quad \frac{\mu}{\lambda+\mu} = \frac{a-\frac{1}{2}(c+d)}{a-b}.$$

Substituting these values in ($\alpha$), it vanishes; hence the truth of the relation.

Again, $\lambda a^2 + \mu b^2 = \text{const.}$ reduces to $(\lambda+\mu)[cd + (\mathrm{CE})^2] = \text{const.}$;

whence $$cd = \frac{\text{const.}}{\lambda+\mu}(\mathrm{CE})^2 = \text{const.}$$

The product of the perpendiculars to L remaining constant, L is a tangent to an ellipse of which C and D are the foci.

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**10028.** (Professor DÉCAMPS.)—On donne sur une droite quatre points A, B, C, D. Sur AC et BD comme cordes, on décrit deux segments capables d'un même angle variable et se coupant en P et Q. Lieu du milieu de corde commune PQ.

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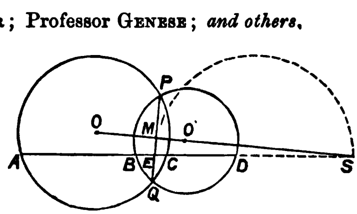
*Solution by* J. C. ST. CLAIR; Professor GENESE; *and others.*

Let O, O′ be the centres of a pair of segments, and let AD meet OO′ in S and PQ in E. Then S is a fixed point, being the ex-centre of similitude of every pair of circles; and, since

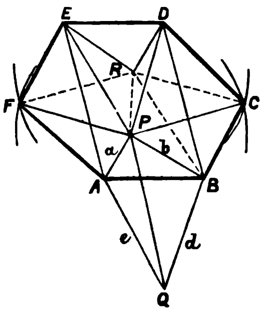



AE. EC = PE. EQ = DE. EB,

the point E is also fixed. Now PQ is bisected at right angles by OO′, therefore EMS is a right angle; hence the locus of M is the circle on ES as diameter.

**10004.** (H. W. Segar.) Given the lengths of six lines supposed to be drawn from any point within a six-sided figure, which is such that its opposite sides are equal and parallel, and also the length of any one side, construct the figure.

*Solution by* H. E. Palliser; G. G. Storr, M.A.; *and others.*




Let $a$, $b$, $c$, $d$, $e$, $f$ be the given straight lines which join the point P within the figure to its angular points A, B, C, &c., and AB the given side. On one side of AB construct a triangle PAB having its sides PA, PB respectively equal to $a$ and $b$, and on the other side of AB construct a triangle QAB having its sides QB, QA respectively equal to $d$ and $e$ (the lines joining P to the extremities of the side of the figure opposite AB). Through A and B draw lines AE, BD equal and parallel to QP. Join ED, PE, PD. Then PE $= e$, PD $= d$, and we see that ED is the side of the figure opposite AB. On ED, on the side not remote from AB, construct a triangle RDE, having its sides RD, RE respectively equal to $a$ and $b$. Then, with centres P and R and radii respectively $c$ and $f$, describe circles cutting in C, and with the same centres and radii, respectively $f$ and $c$, describe circles cutting in F. Join BC, CD, EF, FA. Then ABCDEF is evidently the figure required. For, join PF, PC, RP, RF, RB, RC; then $\angle$ FPR $=$ $\angle$ CRP, and $\angle$ EPR $=$ $\angle$ BRP. Hence $\angle$ FPE $=$ $\angle$ CRB and we have FE $=$ CB, similarly FA $=$ CD. And, since the triangles FEA, CBD are equilateral, they are equiangular, and AE is parallel to BD. Therefore FE, FA are respectively parallel to CB, CD.

[As the order of the vectors is not specified, the first triangle may be constructed in $6!/4! = 30$ ways, or 15 pairs, in each of which one triangle is the image of the other; the second may be constructed in $4!/2! = 12$ ways; and the figure may be completed in two ways. Hence, if the greatest of the lines $a$, $b$, $c$, $d$, $e$, $f$ be less than the sum of the least two, and this sum greater than AB, 720 hexagons may be constructed satisfying the given conditions.]

**10103.** (Professor Sylvester, F.R.S.)—A figure composed of two circular plates of equal diameter in contact is thrown upon a ruled sheet of paper; twice the diameter of either circle being supposed less than the distance between contiguous lines, find the chance of the figure falling in such a position that it will be intersected in four points by one of the ruled lines.

*Solution by* D. Biddle; A. W. Robinson, B.A.; *and others.*

Let radius of circles $=$ unity, and $2a$ $=$ distance between contiguous ruled lines. Also let $x$ $=$ distance of junction of circles from nearest

ruled line, and $\theta$ = angle formed with such line by the line joining the two centres. Then, in all favourable cases, $x$ lies between 0 and 1, and $\theta$ between 0 and $\sin^{-1}(1-x)$. Consequently

$$P = \frac{2}{a\pi}\int_0^1 \sin^{-1} x\, dx = \frac{1}{a}\left(1-\frac{2}{\pi}\right) = (\cdot 3633802)/a.$$

To make the chance exactly one-sixth, the distance between the ruled lines must exceed the combined diameters of the circles in the proportion of about 1·0901407 : 1.

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**9844.** (Professor DE WACHTER.)—Determine a point in the plane of a given ellipse, such that the moment of inertia of the ellipse shall be constant for any *coplanar* axis through that point.

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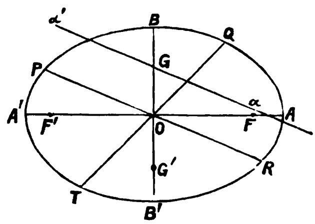
*Solution by the* PROPOSER; Professor SARKAR; *and others.*




The moment of inertia M about any axis $\alpha\alpha'$ at a distance $d$ from a parallel PR drawn through the centre O of the ellipse (axes AA′ = $2a$ and BB′ = $2b$) is $M = \pi abd^2$ + moment of inertia about PR.

The moment of inertia about PR may be derived from that of another ellipse having PR and its conjugate diameter QT for its *rectangular* axes. If we put PR = $2a'$, QT = $2b'$, this moment is expressed by $\frac{1}{4}\pi a'b'^3$, and that of the ellipse, where $\angle$ QOR = $\phi$,

$$= \tfrac{1}{4}\pi a'b'^3 \sin^3\phi = \tfrac{1}{4}\pi ab\,.\,b'^2\sin^2\phi = \pi a^3b^3/4a'^2.$$

Now, $\theta$ being the angle between AA′ and $\alpha\alpha'$, we have

$$a'^2 = a^2\sin^2\theta + b^2\cos^2\theta,$$

and thus the required moment about $\alpha\alpha'$

$$= M = \pi abd^2 + \tfrac{1}{4}\pi ab\,(a^2\sin^2\theta + b^2\cos^2\theta).$$

Calling $p$ and $q$ the coordinates of the required point with reference to OA and OB, the distance $d = p\sin\theta + q\cos\theta$, and thus

$$M = \tfrac{1}{4}\pi ab\left[4\,(p\sin\theta + q\cos\theta)^2 + a^2\sin^2\theta + b^2\cos^2\theta\right],$$

$$= \tfrac{1}{4}\pi ab\left[(4p^2 - 4q^2 + a^2 - b^2)\sin^2\theta + 8pq\cos\theta\sin\theta + 4q^2 + b^2\right].$$

Since M is to be independent of $\theta$, it is necessary to put

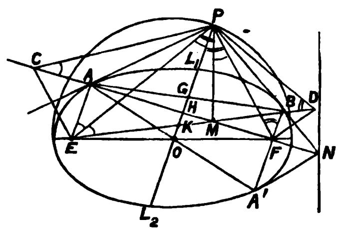
$$4\,(q^2 - p^2) = a^2 - b^2, \text{ and } pq = 0;$$

hence $\quad p = 0;\quad q = \pm\tfrac{1}{2}(a^2-b^2)^{\frac{1}{2}},\quad M = \tfrac{1}{4}\pi a^3b.$

Two points G, G′ answer the question proposed. They lie on the minor axis at equal distances from the centre, and 2GG′ = FF′.

**10118.** (The Editor.)—If from the foci E, F of an ellipse whose centre is O, parallel radii EA, FB be drawn, and tangents AP, BP; prove that (1) OP is parallel to EA; (2) the angles FPA, EPB are right angles; (3) the angle EPF is an arithmetic mean between EAF and EBF; (4) PA . PE : PB . PF = AE : BF; (5) the sum of the reciprocals of EA, FB is constant; (6) the locus of the intersection (M) of the diagonals of the trapezoid AEBF is an ellipse confocal with the given one, and normal at each position to PM; (7) $MP^2 = ME . MF$; (8) PE, PF bisect the angles AEB, AFB; (9) PE . PF = PM (major axis); (10) if through E and F the two other perpendiculars be drawn on the tangents, the sum of the squares of their reciprocals is constant; (11) the locus of P is the circle on the major axis as diameter.

*Solution by* Professor Schoute; G. E. Crawford, B.A.; *and others.*




(1) The diameter OP passes through the mid-point G of the chord AB joining the points where the tangents through P touch the ellipse. Therefore PO is parallel to AE and BF.

(8) This property holds even when A and B are arbitrary points and is generally known (congruency of the triangles CPF and EPD, etc.).

(2) For the mid-point H of AF on OP we have AH = HP = HF, as the angles AFP and PFB or HFP and FPH are equal. This proves APF to be a right angle. And so the mid-point K of EB gives EK = KP = KB, *i.e.*, EPB is a right angle.

(11) The locus of P is the pedal curve of the ellipse for either of the foci, *i.e.*, the circle described on the major axis as diameter.

(3) The angle EPF is equal to the sum of the angles EPO and OPF, *i.e.*, half the sum of the angles AEB and AFB or EBF and EAF.

(4) The triangles APE and BPF, whose angles at P are equal, give PA . PE . sin APE = AE . $l$ and PB . PF . sin FPB = BF . $l$, where $l$ is the distance of O from AE and BF, etc.

(5) In general the sum of the reciprocals of FA′ and FB is constant. This is proved in the easiest manner by Newton's theorem.

We find, calling $OF = c$,

$$\frac{BF\,.\,FA'}{(a+c)(a-c)} = \frac{OL_1^2}{a^2} = \frac{OG\,.\,OP}{a^2} = \frac{OG}{a}, \text{ or } \frac{BF\,.\,FA'}{BF+FA'} = \frac{b^2}{2a},$$

since $\quad OG = \tfrac{1}{2}(BF+FA') \quad$ and $\quad a^2-c^2 = b^2$.

(6) If $AE = pBFq$,

$$EM = EB\,.\,\frac{p}{p+q} = (2a-q)\,\frac{p}{p+q}, \quad MF = (2a-p)\,\frac{q}{p+q},$$

$$EM+MF = 2a - \frac{2pq}{p+q} = 2a - \frac{b^2}{a}.$$

So M describes an ellipse, etc. We find farther

$$HM = \frac{(2a-p)(p-q)}{2(p+q)}, \quad KM = \frac{(2a-q)(p-q)}{2(p+q)},$$

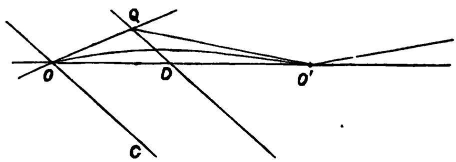
$$HP = a - \tfrac{1}{2}p, \qquad KP = a - \tfrac{1}{2}q;$$

therefore HM : KM = HP : KP, which proves that PM is normal to the ellipse described by M.

(7) The triangles EMP and PMF are similar (for their angles in M are equal and the sum of their angles in P is equal to that of their angles in E and F). This gives $MP^2 = ME \cdot MF$.

(9) The similar triangles EPD and EMP give PD : PM = ED : PE, or $2a \cdot PM = PE \cdot PF$.

(10) In the subjoined Figure, QD is parallel to OC, and D is the mid-point of OO′.




[The last half of (6) may be otherwise proved as follows :—In the triangle MBF, PB bisects the exterior ∠B, and PF the internal ∠F; hence PM bisects external angle M, which shows moreover that P is the ex-centre to both BMF and AME.]

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**9996.** (Hugh MacColl, B.A.)—Show how to calculate the logarithm of any number to any base in a simple and direct manner without any reference to the Napierian base or any other series.

*Solution by* Prof. De Wachter; J. J. Barniville; *and others.*

The following method of approximation, based on the properties of the binary system of numeration, is both elementary and practical; it was devised, I think, by Professor Sarrus, of Strassburg. In

$$A^x = N \quad \ldots\ldots\ldots\ldots\ldots (a),$$

where A and N are positive numbers, the index $x$ may be imagined to be written in the binary scale, thus $x = i \cdot z_1 z_2 z_3 z_4 \ldots$, $i$ being the integer part of $x$, and $z_1$, $z_2$, $z_3$, $z_4$, ... its binary digits (viz. 0 or 1).

1. The value of $i$ is to be determined empirically, which is usually done at first sight. For instance, if $10^{i \cdot z_1 z_2 z_3 z_4 \ldots} = 94385$, $i$ is evidently $= 4$.

2. Next, to find $z_1$, $z_2$, $z_3$, $z_4$, ..., we divide both sides of $(a)$ by $A^i$. This gives

$$A^{0 \cdot z_1 z_2 z_3 z_4 \ldots} = N/A^i = N_1,$$

therefore

$$A^{z_1 \cdot z_2 z_3 z_4 \ldots} = N_1^2 \quad \ldots\ldots\ldots\ldots\ldots (b).$$

For, the squaring of $A^{0 \cdot z_1 z_2 z_3 z_4 \ldots}$ implies duplication of the index. But duplicating a binary number is performed by advancing the binary point by one digit to the right. Now, $z_1$ is either 0 or 1 according to A's being $>$ or $< N_1^2$; thus, $z_1$ being determined, we divide on both sides of $(b)$ by $A^{z_1}$ (which may be $= 1$, viz., if $z_1 = 0$), hence $A^{0 \cdot z_2 z_3 z_4 \ldots} = N_1^2/A^{z_1} = N_2$. Again, $A^{z_2 \cdot z_3 z_4 \ldots} = N_2^2$; which leads to the evaluation of $z_2$. If $A > N_2^2$, $z_2 = 0$; if $A < N_2^2$, $z_2 = 1$.

3. The same process may be extended to any binary digit $z_n$.

Finally, the fraction $0 \cdot z_1 z_2 z_3 z_4 \ldots$ is to be transformed from the binary to the denary scale, which will be done by a table like this:

| Binary Fractions. | Index of Order. | Decimal Values. |
|---|---|---|
| 0·1 | 1 | 0·500000000 |
| 0·01 | 2 | 0·25....... |
| 0·001 | 3 | 0·125...... |
| 0·0001 | 4 | 0·0625..... |
| 0·00001 | 5 | 0·03125.... |
| 0·000001 | 6 | 0·015625... |
| 0·0000001 | 7 | 0·0078125.. |
| 0·00000001 | 8 | 0·00390625. |
| 0·000000001 | 9 | 0·001953125 |
| 0·0000000001 | 10 | 0·0009765625 |
| etc. | etc. | etc. |

**2753.** (Professor Wolstenholme, Sc.D.)—If a straight line be divided at random into four parts, prove that (1) the chance that one of the parts shall be greater than half the line is $\frac{1}{2}$; also the respective chances that (2) three times, and (3) four times, the sum of the squares on the parts, shall be less than the square on the whole line, are $\frac{1}{18}\pi\sqrt{3}$ and $\frac{3}{500}\pi^2\sqrt{5}$.

*Solution by* Professor Schoute.

With reference to a regular tetrahedron, the height of which is equal to the length $l$ of the given line, the four parts $x, y, z, t$ of the line represent the coordinates of a point P situated within the tetrahedron. This point P may be called the image corresponding to the division.

1. When one of the parts is greater than $\frac{1}{2}l$, the image P is situated within one of the four regular tetrahedrons, the vertices of which are formed by the mid-points of three conterminous edges and the point common to these. Each of the four regular tetrahedrons being an eighth part of the tetrahedron of reference, the chance is $\frac{1}{2}$.

2. The condition, that thrice the sum of the squares of the parts is less than $l^2$, is expressed by

$$3(x^2+y^2+z^2+t^2) < (x+y+z+t)^2;$$

and since $$3(x^2+y^2+z^2+t^2)-(x+y+z+t)^2 = 0$$

represents the inscribed sphere, the radius of which is $\frac{1}{4}l$, and the condition is satisfied, when P is situated within the sphere, the chance is

$$W = \tfrac{4}{3}\pi\left(\tfrac{1}{4}l\right)^3 / \left(\tfrac{1}{8}l^3\sqrt{3}\right) = \tfrac{1}{18}\pi\sqrt{3}.$$

3. When four times the sum of the squares of the parts is less than $l^2$, P is situated within the sphere

$$4(x^2+y^2+z^2+t^2)-(x+y+z+t)^2 = 0.$$

But this sphere reduces itself to a point, the centre of the tetrahedron. So we find that the chance is zero instead of $\frac{3}{160}\pi^2\sqrt{5}$.

4. If twice the sum in question is less than $l^2$, P is situated within the sphere that touches the edges of the tetrahedron in their mid-points. Then the chance is the ratio between the area of this sphere diminished by four segments situated without the tetrahedron and the tetrahedron, etc.

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**10072.** (Professor EMMERICH, Ph.D.)—Construct a triangle from the symmedian $AK_a = k_a$, and its distances $p$ and $q$ from B and C.

*Solution by* J. J. BARNIVILLE ; Professor SARKAR ; *and others.*

Draw the symmedian $AK_a$, and on it as hypotenuse construct right-angles with $p$ and $q$ as sides. Then, since $K_a$ is the centroid of PQR, P is given ; draw BC perpendicular to $K_aP$.

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**10109.** (Professor MANNHEIM.)—Une droite se déplace de façon que trois de ses points restent sur trois plans parallèles à une même droite. Démontrer qu'un point de la droite mobile se déplace sur un plan.

*Solution by* Professor SCHOUTE ; J. C. ST. CLAIR ; *and others.*

Si l'on projette la droite mobile $d$ sur un plan $\delta$ perpendiculaire aux trois plans donnés $\alpha$, $\beta$, $\gamma$, il est évident que la projection $d'$ rencontre les traces des trois plans donnés sur $\delta$ en trois points A, B, C, de manière que AB/BC reste constante. En d'autres termes, $d'$ enveloppe une parabole, et un point quelconque D de $d'$, pour lequel le biquotient (ABCD) est constant, parcourt une tangente de cette parabole. Donc, le point de $d$, dont D est la projection, reste dans le plan perpendiculaire à $\delta$ suivant cette tangente, etc.

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**10047.** (MORGAN BRIERLEY.)—Suppose the number of candidates (say 30) at a School Board election be double the number of places to be filled (viz. 15), which latter is also the number of votes any one elector can give to one, or divide amongst two or more candidates. Find in how many different ways the said elector can give his 15 votes. [This Question is said to have caused some excitement, as two different sets of mathematicians have given different solutions, leading to the respective results 614429671, 229916315790.]

**10048.** (Dr. TRAILL. Generalisation of Question 10047.)—If the number of candidates at a School Board election be $n$, and the number of vacancies $r$, which is also the number of votes belonging to any elector, prove that the number of different ways in which the said elector can give his $r$ votes is $(n+r-1)! \div \{r!\,(n-1)!\}$.

[In the particular case which forms Question 10047, this gives $(44!) \div (15!\,29!)$, which agrees with the *second* result in the Note.]

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*Solution by* ANTHONY TRAILL, LL.D.; H. J. WOODALL; *and others.*

The elector can give $r$ votes to a single candidate in $n$ ways. He can select two candidates in $\frac{1}{2}[n(n-1)]$ ways, and he can divide his votes between these two candidates in $(r-1)$ ways. Therefore he can vote for two candidates in $\frac{1}{2}[n(n-1)](r-1)$ ways. He can select three candidates in $n(n-1)(n-2)/(1.2.3)$ ways, and he can divide his votes amongst these three candidates in $\frac{1}{2}(r-1)(r-2)$ ways. Therefore he can vote for three candidates in $n(n-1)(n-2)/(1.2.3)\,[\frac{1}{2}(r-1)(r-2)]$ ways, &c.

Hence the total number (S) of ways in which he can give his $r$ votes is the sum of $r$ terms of the following series

$$S = n + \frac{n.n-1}{1.2}(r-1) + \frac{n.n-1.n-2}{1.2.3}\left(\frac{r-1.r-2}{1.2}\right) + \frac{n.n-1.n-2.n-3}{1.2.3.4}\left(\frac{r-1.r-2.r-3}{1.2.3}\right) + \&c.$$

But S is evidently the coefficient of $x^r$ in the expansion of

$$\left[(1+x)^n-1\right]\left[(x+1)^{r-1}\right],$$

or

$$\left[nx + \frac{n.n-1}{1.2}x^2 + \frac{n.n-1.n-2}{1.2.3}x^3 + \&c.\right] \times \left[x^{r-1} + (r-1)x^{r-2} + \frac{r-1.r-2}{1.2}x^{r-3} + \&c.\right].$$

But $\left[(1+x)^n-1\right]\left[(1+x)^{r-1}\right] = (1+x)^{n+r-1} - (1+x)^{r-1}$,

and the latter binomial does not contain $x^r$, therefore S is the coefficient of $x^r$ in the expansion of $(1+x)^{n+r-1}$, viz.,

$$\frac{(n+r-1)(n+r-2)\dots(n+1)\,n}{1.2.3\dots r} \text{ or } \frac{(n+r-1)!}{r!\,(n-1)!}.$$

But this number also represents the number of combinations of $(n+r-1)$ things taken $r$ at a time.

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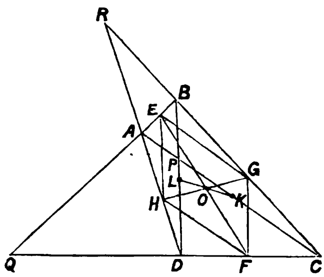
**10057.** (E. CÉSARO.)—Les coniques issues de P, osculant en un point Q une même ligne, ont leurs centres sur une conique dont le centre est aux trois quarts de PQ.

*Solution by* Professor SCHOUTE ; J. C. ST. CLAIR ; *and others.*

The locus of the centre of the conics that pass through the four points A, B, C, D is a conic, nine points of which are easily indicated—the mid-points E, F, G, H, K, L, of the six sides, and the diagonal points P, Q, R of the complete quadrangle ABCD. In the case of the equilateral hyperbolas that pass through the vertices of a triangle ABC (and its orthocentre), this conic with the nine points is the classic *circle of the nine points.*




The centre of gravity of the four points A, B, C, D (equally charged) is the mid-point O of the segments EF, GH, KL, and therefore the centre of the conic through these six points. For two chords of a conic, that bisect one another, pass through its centre. Therefore, when the points A, B, C are three consecutive points of a given curve, the centre of the conic O lies on AD, and so as to make AO = $\frac{1}{3}$OD.

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**10119.** (HUGH MACCOLL, B.A.)—
To put a question somewhat queer, Suppose the earth an *airless sphere,*
Through which is bored, from pole to pole, A bottomless cylindric hole;
Right o'er the centre let there fall A smooth and polished marble ball.
Dear Reader, may I ask you *when* That marble would come back again?
[See the PROPOSER's amusing story, *Mr. Stranger's Sealed Packet.*]

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*Solution by* D. BIDDLE ; J. C. ST. CLAIR ; *and others.*

Let the sphere (of radius = 1) be supposed to consist of concentric shells; it is easy to prove that those to which the marble is internal, at a distance $(1-x)$ from the centre, exert no attractive force upon it in one direction more than another (*vide* Professor TAIT's *Properties of Matter*, p. 113). Those to which it is external attract it with a force directly proportional to their mass, and inversely proportional to the square of the distance of the marble from their common centre. Therefore the attractive force varies as $(1-x)^3/(1-x)^2 = 1-x$.

One of the ordinary formulæ in regard to falling bodies is $v^2 = 2fs$, in which $v \equiv$ velocity, $s \equiv$ distance passed through from rest, and $f \equiv$ force, assumed to be identical throughout, which however it never is. This formula is derived from $\int_0^v z\,dz = f\int_0^s dx$. But it is easy to see that in the present case $\int_0^v z\,dz = f\int_0^s (1-x)\,dx$, whence $v^2 = 2fs - fs^2$. Now the time occupied in falling through an infinitesimal distance is inversely pro-

portional to the velocity. Consequently, the time occupied in reaching the centre of the sphere is given by

$$t = \int_0^1 \frac{1}{(2fx - fx^2)^{\frac{1}{2}}}\,dx = \left(\frac{1}{f}\right)^{\frac{1}{2}} 2\sin^{-1}\left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{\pi}{2}\left(\frac{1}{f}\right)^{\frac{1}{2}}.$$

The value of $f$ (taken in reference to $R = 1$) may be found as follows: The force of gravity, at sea-level in any latitude $\lambda$ on the spheroidal earth, $= 32\cdot088\,(1 + 0\cdot00513 \sin^2\lambda)$. This gives, at the pole, $32\cdot2526$. But as the earth is now to be regarded as a sphere, allowance must be made. According to Col. CLARKE, the mean semi-axes of the earth are in feet 20926202 and 20854895 (*vide Geodesy*, p. 319), which yield for a *sphere* of equal volume a radius of 20902410 feet; whence, by the law of inverse squares, the initial force of gravity at the pole becomes $32\cdot10614$, or $R/651041$; therefore $\left(\frac{1}{f}\right)^{\frac{1}{2}} = 806\cdot871$ and $\frac{\pi}{2}\left(\frac{1}{f}\right)^{\frac{1}{2}} = 1267\cdot43$ seconds. The marble will rise to the further extremity of the cylindrical hole in the same time, and then return, the total time occupied before it re-appears being $4 \times \frac{\pi}{2}\left(\frac{1}{f}\right)^{\frac{1}{2}} = 5069\cdot72$ seconds $= 1$ hour 24 minutes $29\cdot7$ seconds. Its velocity in passing the centre is given by $V = R\,(2fs - fs^2)^{\frac{1}{2}}$, when $s = 1$; namely, $25{,}905\cdot5$ feet per second.

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**9697.** (Professor WOLSTENHOLME, M.A., Sc.D. Extension of Quest. 9430, for which see Vol. XLIX., p. 88.)—From any equation of the form $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$ may be deduced $A_1 \pm A_2 \pm A_3 \pm A_4 = 0$, where $A_1$, $A_2$, $A_3$, $A_4$ are the areas of the faces. This equation when rationalised is of course symmetrical in 1, 2, 3, 4; hence the equations $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$, $\beta_1 + \beta_2 = \beta_3 + \beta_4$, $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ are not independent. When the system of equations $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$, $\beta_1 + \beta_2 = \beta_3 + \beta_4$, $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ holds, the equation between the areas of the faces is $A_1 + A_2 = A_3 + A_4$. With the notation of Quest. 9430, and denoting by $S_1$, $S_2$, $S_3$, $S_4$ the areas of the triangular faces, by $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ the semi-sums of the plane angles at the corners of the tetrahedron, the system of equations

$$\begin{aligned} &\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4, \quad \beta_1 + \beta_2 = \beta_3 + \beta_4, \quad \gamma_1 + \gamma_2 = \gamma_3 + \gamma_4, \\ &\sigma_1 + \sigma_2 = \sigma_3 + \sigma_4, \quad \sigma_1 - \beta_1 = \sigma_4 - \beta_4, \quad \sigma_1 - \gamma_1 = \sigma_3 - \gamma_3, \\ &\sigma_2 - \beta_2 = \sigma_3 - \beta_3, \quad \sigma_2 - \gamma_2 = \sigma_4 - \gamma_4, \quad S_1 + S_2 = S_3 + S_4, \end{aligned}$$

is such that, if any one be true, every one of the system is true.

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*Solution by the* PROPOSER.

In a tetrahedron OABC, let the lengths OA, OB, OC be denoted by $a$, $b$, $c$, and the lengths BC, CA, AB by $x$, $y$, $z$; the angles subtended by BC at O, A by $\alpha_1$, $\alpha_2$, those subtended by OA at B, C by $\alpha_3$, $\alpha_4$; and similarly for $\beta$, $\gamma$. Then

$$2bc\cos\alpha_1 = b^2 + c^2 - x^2, \quad bc\sin\alpha_1 = 2S_2,$$
$$2yz\cos\alpha_2 = y^2 + z^2 - x^2, \quad yz\sin\alpha_2 = 2S_1,$$

so that $$4bcyz\cos(\alpha_1+\alpha_2) = (b^2+c^2-x^2)(y^2+z^2-x^2)-16S_1S_2,$$

Similarly, $$4bcyz\cos(\alpha_3+\alpha_4) = (b^2+y^2-a^2)(c^2+z^2-a^2)-16S_3S_4,$$

so that $$4bcyz\,[\cos(\alpha_3+\alpha_4)-\cos(\alpha_1+\alpha_2) = (b^2-y^2)(c^2-z^2)$$
$$-(a^2-x^2)(b^2+c^2+y^2+z^2-a^2-x^2)+16S_1S_2-16S_3S_4.$$

Now, $$16(S_1^2-S_4^2) = 2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4$$
$$-2a^2b^2-2a^2z^2-2b^2z^2+a^4+b^4+z^4$$
$$\equiv (a^2-b^2)^2-(y^2-z^2)^2-2z^2(a^2+b^2-x^2-y^2),$$

and $$16(S_2^2-S_3^2) = 2b^2c^2+2b^2x^2+2c^2x^2-x^4-b^4-c^4$$
$$-2a^2c^2-2a^2y^2-2c^2y^2+a^4+c^4+y^4$$
$$\equiv (a^2-y^2)^2-(b^2-x^2)^2+2c^2(b^2+x^2-a^2-y^2);$$

whence $$8(S_1^2+S_2^2-S_3^2-S_4^2) = (b^2-y^2)(c^2-z^2)$$
$$-(a^2-x^2)(b^2+c^2+y^2+z^2-a^2-x^2),$$

and $$bcyz\,[\cos(\alpha_3+\alpha_4)-\cos(\alpha_1+\alpha_2) = 2\,[(S_1+S_2)^2-(S_3+S_4)^2],$$

or $$bcyz\sin\tfrac{1}{2}(\alpha_1+\alpha_2-\alpha_3-\alpha_4)\sin\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$
$$= (S_1+S_2-S_3-S_4)(S_1+S_2+S_3+S_4).$$

Now the factors $\sin\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, $S_1+S_2+S_3+S_4$ are both always finite (and positive) in any real finite tetrahedron. Hence the equations $\alpha_1+\alpha_2=\alpha_3+\alpha_4$, $S_1+S_2=S_3+S_4$ are equivalent to each other. Similar equations may be now written down in $\beta$ and $\gamma$ instead of $\alpha$, from which it follows that the equations

$$\alpha_1+\alpha_2=\alpha_3+\alpha_4,\ \beta_1+\beta_2=\beta_3+\beta_4,\ \gamma_1+\gamma_2=\gamma_3+\gamma_4,\ S_1+S_2=S_3+S_4$$

are such that, when any one is true, all are true; as are also

$$\sigma_1+\sigma_2=\pi=\sigma_3+\sigma_4;$$

and, remembering that

$$\alpha_2+\beta_3+\gamma_4=\pi=\alpha_1+\beta_4+\gamma_3=\alpha_4+\beta_1+\gamma_2=\alpha_3+\beta_2+\gamma_1,$$

it is easy to deduce the system

$$\sigma_1-\gamma_1=\sigma_3-\gamma_3,\ \sigma_1-\beta_1=\sigma_4-\beta_4,\ \sigma_2-\beta_2=\sigma_3-\beta_3,\ \sigma_2-\gamma_2=\sigma_4-\gamma_4,$$

and, if in any tetrahedron any one of all these equations hold, the whole system will hold.

It appears from the above that, in any finite tetrahedron,

$$(1/ax)\,[\sin\tfrac{1}{2}(\alpha_1+\alpha_2-\alpha_3-\alpha_4)\sin\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)]$$
$$=(1/by)\,[\sin\tfrac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)\sin\tfrac{1}{2}(\beta_1+\beta_2+\beta_3+\beta_4)]$$
$$=(1/cz)\,[\sin\tfrac{1}{2}(\gamma_1+\gamma_2-\gamma_3-\gamma_4)\sin\tfrac{1}{2}(\gamma_1+\gamma_2+\gamma_3+\gamma_4)]$$
$$=\frac{(S_1+S_2-S_3-S_4)(S_1+S_2-S_3-S_4)}{abcxyz},$$

also that

$$\frac{\sin\frac{1}{2}(\alpha_1+\alpha_2-\alpha_3-\alpha_4)}{S_1+S_2-S_3-S_4}=\frac{\sin\frac{1}{2}(\alpha_3+\alpha_1-\alpha_2-\alpha_4)}{S_3+S_1-S_2-S_4}=\frac{\sin\frac{1}{2}(\alpha_2+\alpha_3-\alpha_1-\alpha_4)}{S_2+S_3-S_1-S_4},$$

with similar equations in $\beta$, $\gamma$. These are also all positive, so that, if $S_1+S_2>S_3+S_4$, $\alpha_1+\alpha_2>\alpha_3+\alpha_4$, $\beta_1+\beta_2>\beta_3+\beta_4$, and $\gamma_1+\gamma_2>\gamma_3+\gamma_4$.

**9989.** (Professor ABINASH CHANDRA BASÛ.)—Prove (1) that

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, \quad 3Ax^4 + 2Bx^3 + Cx^2 - E = 0 \ldots\ldots\ldots\ldots (\alpha, \beta)$$

are so related that, if they have a common root, that root will be a double root of $(\alpha)$; (2) if the roots of $(\alpha)$ be $a$, $b$, $c$, $d$, show how to express the latter in the form of a determinant of the fourth order.

*Solution by* JOHN J. BARNIVILLE.

The first derived function $\int^2 (x)$ of $(\alpha)$ is $3Ax^3 + 2Bx^2 + Cx + d$; thus, if $(\alpha)$ and $(\beta)$ have a common root, $(\alpha)$ and $\int^2 (x) = 0$ will have the same common root, consequently $(\alpha)$ will have two roots equal to it.

**10055.** (A. SOYER.)—Résoudre le système d'équations

$$(x+y)(1+xy+x^2y+xy^2+x^2y^2)+xy = a,$$
$$xy(x+y)(x+y+xy)(x+y+xy+x^2y+xy^2) = b.$$

*Solution by* L. WIENER; J. J. BARNIVILLE; *and others.*

Substituting $z$ for $x+y$, and $t$ for $xy$, the equations become

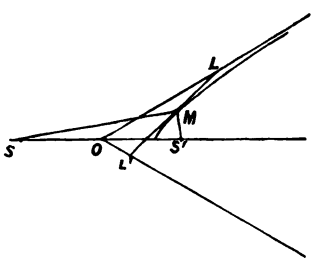
$$z+t+zt+z^2t+zt^2 = a, \quad zt(z+t)(z+t+zt) = b;$$

substituting $f$ for $zt+$, and $g$ for $zt$, we get a new set of equations $f+g+fg = a$, and $fg(f+g) = b$, and again substituting $l$ for $f+g$ and $m$ for $fg$, we have $l+m = a$, and $lm = b$, which last equation is easily solved; then we find $f$ and $g$, next $z$ and $t$, and at last the values for $x$ and $y$.

**10032.** (Professor GENESE, M.A.)—S, S′ are the foci of a conic LL′, a tangent terminated by the asymptotes. Prove that L, L′ are the foci of a conic of which SS′ is a tangent terminated by the asymptotes.

*Solution by* L. WIENER; R. H. W. WHAPHAM, B.A.; *and others.*

The hyperbola, the foci of which are S and S′, and that has OL and OL′ for asymptotes, touches LL′ at M, half-way between L and L′, and LL′ bisects the angle SMS′; hence L and L′ are the foci of a new hyperbola of which MS and MS′ are the asymptotes, and which touches SS′ at O, half-way between S, S′. It is evident that no other conic will answer the problem, since LL′ is given in length as cut off by asymptotes.



**9977.** (Professor LAMPE.)—Investigate formulæ for the sums of the powers of the rational numbers, and, with the notation $S_k(x) = 1^k + 2^k + \ldots + x^k$, prove, from general expressions, that

$$16S_1^5 = S_5 + 10S_7 + 5S_9, \quad 12S_2^3 = S_4 + 7S_6 + 4S_8, \quad 12S_1S_5 = -S_3 + 5S_5 + 8S_7,$$
$$30S_2S_4 = -S_3 + 15S_5 + 16S_7, \quad 72S_1S_2S_3 = 3S_4 + 42S_6 + 27S_8.$$

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*Solution by* (1) J. D. H. DICKSON *and* J. J. BARNIVILLE; (2) *the* PROPOSER.

1. In Vol. XLIX., p. 179, the following equations are given as examples of a general formula there quoted, viz.:

$$5S_4 = 3S_2\left\{2S_1 - \tfrac{1}{3}\right\}, \qquad 8S_7 = 4S_3\left\{4S_3 - \tfrac{8}{3}S_1 + \tfrac{2}{3}\right\},$$
$$6S_5 = 4S_3\left\{2S_1 - \tfrac{1}{2}\right\}, \qquad 9S_8 = 3S_2\left\{6S_5 - 6S_3 + \tfrac{18}{5}S_1 - \tfrac{3}{5}\right\},$$
$$7S_6 = 3S_2\left\{4S_3 - 2S_1 + \tfrac{1}{3}\right\}, \qquad 10S_9 = 4S_3\left\{6S_5 - 8S_3 + 6S_1 - \tfrac{3}{2}\right\}.$$

On substituting in the equations to be proved, they become identities.

2. *Otherwise*:—In 1877 I published in the *Journal für die reine und angewandte Mathematik* [usually cited in English as CRELLE's *Journal*], Vol. LXXXIV., pp. 270–272, a little note concerning the sums

$$S_k(x) = 1^k + 2^k + 3^k + \ldots + x^k,$$

and revealing the source of a great many relations between them. I then did not think it necessary to enter into details, because the general symbolical formula which I gave furnishes in an easy way, by mechanical computations, several series of all desirable relations. The questions recently proposed in the *Educational Times* and in the *Reprints* called this note of mine to my mind, and I avail myself of this occasion to give some further developments of the object.

Prof. M. STERN had inserted in the same volume of the *Journal*, p. 216, a note entitled, "Verallgemeinerung einer Jacobischen Formel," in which he proved, by the Bernoullian method of induction, the elegant formula—

$$\text{I.} \qquad 2^{n-1} S_1^n = \binom{n}{1} S_{2n-1} + \binom{n}{3} S_{2n-3} + \binom{n}{5} S_{2n-5} + \ldots$$

Whereupon I gave a direct proof of all formulæ of similar character. On account of its elementary character, I deem it advisable to illustrate it with this same example. We have

$$S_1(x) = 1 + 2 + 3 + \ldots + x = \tfrac{1}{2}x(x+1),$$
$$S_1(x-1) = 1 + 2 + 3 + \ldots + (x-1) = \tfrac{1}{2}x(x-1).$$

Therefore $\qquad 2^n\left[S_1(x)^n - S_1(x-1)^n\right] = x^n\left[(x+1)^n - (x-1)^n\right],$

or $\quad 2^{n-1}\left[S_1(x)^n - S_1(x-1)^n\right] = \binom{n}{1} x^{2n-1} + \binom{n}{3} x^{2n-3} + \binom{n}{5} x^{2n-5} + \ldots.$

Substituting successively 1, 2, 3, ..., $x$ for $x$, we get—

$$(1)\ 2^{n-1}[S_1(1)^n - S_1(0)^n] = \binom{n}{1} 1^{2n-1} + \binom{n}{3} 1^{2n-3} + \binom{n}{5} 1^{2n-5} + \dots,$$

$$(2)\ 2^{n-1}[S_1(2)^n - S_1(1)^n] = \binom{n}{1} 2^{2n-1} + \binom{n}{3} 2^{2n-3} + \binom{n}{5} 2^{2n-5} + \dots,$$

$$(3)\ 2^{n-1}[S_1(3)^n - S_1(2)^n] = \binom{n}{1} 3^{2n-1} + \binom{n}{3} 3^{2n-3} + \binom{n}{5} 3^{2n-5} + \dots,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$(x)\ 2^{n-1}[S_1(x)^n - S_1(x-1)^n] = \binom{n}{1} x^{2n-1} + \binom{n}{3} x^{2n-3} + \binom{n}{5} x^{2n-5}.$$

Adding these $x$ equations, we evidently obtain the relation I., containing as special cases—

$$S_1^2 = S_3, \qquad 16S_1^5 = S_5 + 10S_7 + 5S_9,$$

$$4S_1^3 = S_3 + 3S_5, \qquad 16S_1^6 = 3S_7 + 10S_9 + 3S_{11},$$

$$2S_1^4 = S_5 + S_7, \qquad 64S_1^7 = S_7 + 21S_9 + 35S_{11} + 7S_{13}, \text{ \&c.}$$

In the same way we come to the formula for the powers of

$$S_2(x) = \tfrac{1}{6}x(x+1)(2x+1),$$

$$\text{II. } \tfrac{1}{3}.6^n S_2^n = 2^{n-1}.3\binom{n}{1}S_{3n-1} + 2^{n-3}\left\{(2^3+1)\binom{n}{3} + (2^2+2)\binom{n}{2}\binom{n}{1}\right\}S_{3n-3}$$

$$+ 2^{n-5}\left\{(2^5+1)\binom{n}{5} + (2^4+2)\binom{n}{4}\binom{n}{1} + (2^3+2^2)\binom{n}{3}\binom{n}{2}\right\}S_{3n-5}$$

$$+ 2^{n-7}\left\{(2^7+1)\binom{n}{7} + (2^6+2)\binom{n}{6}\binom{n}{1} + (2^5+2^2)\binom{n}{5}\binom{n}{2} + (2^4+2^3)\binom{n}{4}\binom{n}{3}\right\}S_{3n-7} + \dots,$$

with the following examples for $n = 2, 3, 4, 5$:

$$3S_2^2 = S_3 + 2S_5,$$

$$12S_2^3 = S_4 + 7S_6 + 4S_8,$$

$$54S_2^4 = S_5 + 15S_7 + 30S_9 + 8S_{11},$$

$$1296S_2^5 = 5S_6 + 130S_8 + 561S_{10} + 520S_{12} + 80S_{14}.$$

In order to obtain the general formula for $S_i(x)^n$ we recur to the expression for $S_i(x)$ given by JAMES BERNOULLI in his *Ars conjectandi*,

$$S_i(x) = \frac{1}{i+1}x^{i+1} + \tfrac{1}{2}x^i + \tfrac{1}{2}B_1\binom{i}{1}x^{i-1} - \tfrac{1}{4}B_2\binom{i}{3}x^{i-3} + \tfrac{1}{6}B_3\binom{i}{5}x^{i-5} - \dots$$

where $B_p$ is the $p^{\text{th}}$ number of Bernoulli, and $\binom{i}{i}$ is to be supposed $= 0$

(not $=1$ as commonly). Subtracting $x^i$, it follows—

$$S_i(x-1)=\frac{1}{i+1}x^{i+1}-\tfrac{1}{2}x^i+\tfrac{1}{2}B_1\binom{i}{1}x^{i-1}-\tfrac{1}{4}B_2\binom{i}{3}x^{i-3}+\tfrac{1}{6}B_3\binom{i}{5}x^{i-5}-\dots.$$

Forming $S_i(x)^n-S_i(x-1)^n$, we find it to be

$$\left\{\frac{1}{i+1}x^{i+1}+\tfrac{1}{2}x^i+\tfrac{1}{2}B_1\binom{i}{1}x^{i-1}-\tfrac{1}{4}B_2\binom{i}{3}x^{i-3}+\dots\right\}$$
$$-\left\{\frac{1}{i+1}x^{i+1}-\tfrac{1}{2}x^i+\tfrac{1}{2}B_1\binom{i}{1}x^{i-1}-\tfrac{1}{4}B_2\binom{i}{3}x^{i-3}+\dots\right\}.$$

Expanding and treating this equality just as in the two preceding cases for $i=1$ and $i=2$, we may write the result in the following symbolical form (p. 271 of my former publication):—

$$\text{III.}\quad S_i^n=\left\{\frac{1}{i+1}S^{i+1}+\tfrac{1}{2}S^i+\tfrac{1}{2}B_1\binom{i}{1}S^{i-1}-\tfrac{1}{4}B_2\binom{i}{3}S^{i-3}+\tfrac{1}{6}B_3\binom{i}{5}S^{i-5}-\dots\right\}^n$$
$$-\left\{\frac{1}{i+1}S^{i+1}-\tfrac{1}{2}S^i+\tfrac{1}{2}B_1\binom{i}{1}S^{i-1}-\tfrac{1}{4}B_2\binom{i}{3}S^{i-3}+\tfrac{1}{6}B_3\binom{i}{5}S^{i-5}-\dots\right\}^n.$$

The $n^{\text{th}}$ powers of the polynomial expressions are to be computed, and then the exponents of all S must be put down as indices.

In the same way the product

IV. $S_iS_kS_l\dots$

$$=\left(\frac{1}{i+1}S^{i+1}+\tfrac{1}{2}S^i+\dots\right)\left(\frac{1}{k+1}S^{k+1}+\tfrac{1}{2}S^k+\dots\right)\left(\frac{1}{l+1}S^{l+1}+\tfrac{1}{2}S^l+\dots\right)$$
$$-\left(\frac{1}{i+1}S^{i+1}-\tfrac{1}{2}S^i+\dots\right)\left(\frac{1}{k+1}S^{k+1}-\tfrac{1}{2}S^k+\dots\right)\left(\frac{1}{l+1}S^l-\tfrac{1}{2}S^l+\dots\right)$$

is expressed as a sum of the quantities $S_m$.

As an illustration of IV. let us develop $S_i\,.\,S_k$. Putting

$$A_i=\frac{1}{i+1}x^{i+1}+\tfrac{1}{2}B_1\binom{i}{1}x^{i-1}-\tfrac{1}{4}B_2\binom{i}{3}x^{i-3}+\tfrac{1}{6}B_3\binom{i}{5}x^{i-5}-\dots,$$

we have $\qquad S_i(x)=A_i+\tfrac{1}{2}x^i,\quad S_i(x-1)=A_i-\tfrac{1}{2}x^i,$

therefore $S_i(x)\,S_k(x)-S_i(x-1)\,S_k(x-1)=(A_i+\tfrac{1}{2}x^i)(A_k+\tfrac{1}{2}x^k)$
$$-(A_i-\tfrac{1}{2}x^i)(A_k-\tfrac{1}{2}x^k)=A_ix^k+A_kx^i$$
$$=\left(\frac{1}{i+1}+\frac{1}{k+1}\right)x^{i+k+1}+\tfrac{1}{2}B_1\left\{\binom{i}{1}+\binom{k}{1}\right\}x^{i+k-1}$$
$$-\tfrac{1}{4}B_2\left\{\binom{i}{3}+\binom{k}{3}\right\}x^{i+k-3}+\dots;$$

whence

$$\text{V.}\quad S_iS_k=\left(\frac{1}{i+1}+\frac{1}{k+1}\right)S_{i+k+1}+\tfrac{1}{2}B_1\left\{\binom{i}{1}+\binom{k}{1}\right\}S_{i+k-1}$$
$$-\tfrac{1}{4}B_2\left\{\binom{i}{3}+\binom{k}{3}\right\}S_{i+k-3}+\tfrac{1}{6}B_3\left\{\binom{i}{5}+\binom{k}{5}\right\}S_{i+k-5}-\dots;$$

and for $i = k$,

$$\text{VI. } \tfrac{1}{2}S_i^2 = \frac{1}{i+1} S_{2i+1} + \tfrac{1}{2}B_1 \binom{i}{1} S_{2i-1}$$
$$- \tfrac{1}{4}B_2 \binom{i}{3} S_{2i-3} + \tfrac{1}{6}B_3 \binom{i}{5} S_{2i-5} - \ldots$$

We must bear in mind that in V. and VI. $\binom{i}{i}$ is to be taken as 0.

Examples for VI. ($B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, &c.):

$$\begin{aligned}
\tfrac{1}{2}S_1^2 &= \tfrac{1}{2}S_3, \\
\tfrac{1}{2}S_2^2 &= \tfrac{1}{3}S_5 + \tfrac{1}{2}B_1 \tbinom{2}{1} S_3, \\
\tfrac{1}{2}S_3^2 &= \tfrac{1}{4}S_7 + \tfrac{1}{2}B_1 \tbinom{3}{1} S_5, \\
\tfrac{1}{2}S_4^2 &= \tfrac{1}{5}S_9 + \tfrac{1}{2}B_1 \tbinom{4}{1} S_7 - \tfrac{1}{4}B_2 \tbinom{4}{3} S_5, \\
\tfrac{1}{2}S_5^2 &= \tfrac{1}{6}S_{11} + \tfrac{1}{2}B_1 \tbinom{5}{1} S_9 - \tfrac{1}{4}B_2 \tbinom{5}{3} S_7, \\
\tfrac{1}{2}S_6^2 &= \tfrac{1}{7}S_{13} + \tfrac{1}{2}B_1 \tbinom{6}{1} S_{11} - \tfrac{1}{4}B_2 \tbinom{6}{3} S_9 + \tfrac{1}{6}B_3 \tbinom{6}{5} S_7, \\
\tfrac{1}{2}S_7^2 &= \tfrac{1}{8}S_{15} + \tfrac{1}{2}B_1 \tbinom{7}{1} S_{13} - \tfrac{1}{4}B_2 \tbinom{7}{3} S_{11} + \tfrac{1}{6}B_3 \tbinom{7}{5} S_9. \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}$$

Examples for V.:

$$\begin{aligned}
S_1S_2 &= (\tfrac{1}{2} + \tfrac{1}{3}) S_4 + \tfrac{1}{2}B_1 \tbinom{2}{1} S_2, \\
S_1S_3 &= (\tfrac{1}{2} + \tfrac{1}{4}) S_5 + \tfrac{1}{2}B_1 \tbinom{3}{1} S_3, \\
S_1S_4 &= (\tfrac{1}{2} + \tfrac{1}{5}) S_6 + \tfrac{1}{2}B_1 \tbinom{4}{1} S_4 - \tfrac{1}{4}B_2 \tbinom{4}{3} S_2, \\
S_1S_5 &= (\tfrac{1}{2} + \tfrac{1}{6}) S_7 + \tfrac{1}{2}B_1 \tbinom{5}{1} S_5 - \tfrac{1}{4}B_2 \tbinom{5}{3} S_3, \\
S_1S_6 &= (\tfrac{1}{2} + \tfrac{1}{7}) S_8 + \tfrac{1}{2}B_1 \tbinom{6}{1} S_6 - \tfrac{1}{4}B_2 \tbinom{6}{3} S_4 + \tfrac{1}{6}B_3 \tbinom{6}{5} S_2, \\
S_1S_7 &= (\tfrac{1}{2} + \tfrac{1}{8}) S_9 + \tfrac{1}{2}B_1 \tbinom{7}{1} S_7 - \tfrac{1}{4}B_2 \tbinom{7}{3} S_5 + \tfrac{1}{6}B_3 \tbinom{7}{5} S_3, \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
S_2S_3 &= (\tfrac{1}{3} + \tfrac{1}{4}) S_6 + \tfrac{1}{2}B_1 \left\{\tbinom{2}{1} + \tbinom{3}{1}\right\} S_4, \\
S_2S_4 &= (\tfrac{1}{3} + \tfrac{1}{5}) S_7 + \tfrac{1}{2}B_1 \left\{\tbinom{2}{1} + \tbinom{4}{1}\right\} S_5 - \tfrac{1}{4}B_2 \tbinom{4}{3} S_3, \\
S_2S_5 &= (\tfrac{1}{3} + \tfrac{1}{6}) S_8 + \tfrac{1}{2}B_1 \left\{\tbinom{2}{1} + \tbinom{5}{1}\right\} S_6 - \tfrac{1}{4}B_2 \tbinom{5}{3} S_4, \\
S_2S_6 &= (\tfrac{1}{3} + \tfrac{1}{7}) S_9 + \tfrac{1}{2}B_1 \left\{\tbinom{2}{1} + \tbinom{6}{1}\right\} S_7 - \tfrac{1}{4}B_2 \tbinom{6}{3} S_5 + \tfrac{1}{6}B_3 \tbinom{6}{5} S_3, \\
S_2S_7 &= (\tfrac{1}{3} + \tfrac{1}{8}) S_{10} + \tfrac{1}{2}B_1 \left\{\tbinom{2}{1} + \tbinom{7}{1}\right\} S_8 - \tfrac{1}{4}B_2 \tbinom{7}{3} S_6 + \tfrac{1}{6}B_3 \tbinom{7}{5} S_4, \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}$$

Making $x = 1$ in these equations, all S become unity, and thus we get as many sets of equations for the Bernoullian numbers B which evaluate immediately in forms bearing an evident relation to their composition of the fractions $1/n$. Besides, a convenient elimination furnishes new formulæ, and mere intuition shows the existence of certain factors like $S_1^2$ or $S_2$.

[Owing to lack of space, we regret that we cannot find room for the many further examples—though interesting and valuable—which have been forwarded to us by Dr. Lampe.]

**2683.** (R. Tucker, M.A.)—To each point on the circumscribing circle of a triangle corresponds a foot-perpendicular line; this cuts the circle in two points; required the locus of the intersection of the feet-perpendicular lines corresponding to these points of section.

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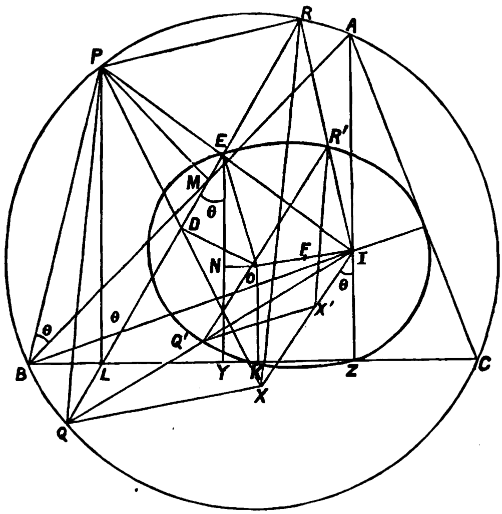
*Solution by* G. E. Crawford, B.A.

I assume that the following construction for the pedal line is known:—Join P to the orthocentre I of the triangle ABC, and through E the mid-point of IP, draw a straight line making with AI the same angle that PA subtends at the circumference of the circle.




[*** In the diagram, the inner curve is meant to be the 9-point circle, with centre F. This circle passes correctly through E, R′, Z, K, Q′, but its shape is somewhat deformed.]

Now in the triangle ABC let I be the ortho-, O the circum-, and F the nine-point centre, and QR the pedal line of P.

Then the pedal line of R passes through R′, the mid-point of IR, and is inclined to AI at an angle measured by RA.

But angle RBA = PBA − PBR = PLE − PQR (since PBLM are concyclic) = QPL, therefore PQ and PR give the directions of the pedals of R and Q. So, drawing R′X′, Q′X′ parallel to PQ, PR, we require to find the locus of X′. Produce IX′ to X, so that IX = 2IX′. Join RX, OX, PX; then—RX, QX, being parallel to R′X′, Q′X′, and therefore to PQ, PR—PX, QR must bisect each other in D, and IX be parallel to ED.

Join OD bisecting QR at right angles, and draw EY, OK perpen-

dicular to BC, and ON perpendicular to EY. Now angle XIZ = PLR by parallels = PBA = $\theta$, say. And, using polar coordinates,

$$r = \text{IX}' = \tfrac{1}{2}\text{IX} = \text{DE} = \text{EN}\cos\theta - \text{ON}\sin\theta \text{ by projecting}$$
$$= \cos\theta\left\{\tfrac{1}{2}(\text{PL}+\text{IZ})-\text{NY}\right\}-\sin\theta\,(\text{LK}-\text{LY}).$$

But $\text{PL} = \text{PB}\sin(\text{B}+\theta) = 2a\sin\text{PCB}\sin(\text{B}+\theta)$ if $a$ = radius OP
$$= 2a\sin(\text{C}-\theta)\sin(\text{B}+\theta).$$
Similarly $\quad \text{BL} = 2a\sin(\text{C}-\theta)\cos(\text{B}+\theta),$

and $\quad \text{LK}-\text{LY} = (\text{BK}-\text{BL})-\tfrac{1}{2}(\text{BZ}-\text{BL}) = \text{BK}-\tfrac{1}{2}\text{BZ}-\tfrac{1}{2}\text{BL}$
$$= \text{BK}-\tfrac{1}{2}\text{BZ}-a\sin(\text{C}-\theta)\cos(\text{B}+\theta).$$

Hence $\quad r = \cos\theta\left\{a\sin(\text{C}-\theta)\sin(\text{B}+\theta)+a\cos\text{B}\cos\text{C}-a\cos\text{A}\right\}$
$$+\sin\theta\left\{a\sin\text{C}\cos\text{B}+a\sin(\text{C}-\theta)\cos(\text{B}+\theta)-a\sin\text{A}\right\}$$
$$= 2a\cos(\text{C}-\theta)\cos(\text{B}+\theta)\cos\theta, \text{ on reducing,}$$

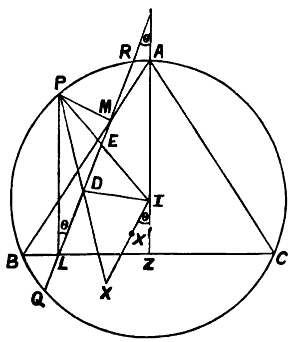
which is the required locus.

Putting $\theta = 0$, we see the locus passes through Z (for IZ = $2a\cos\text{C}\cos\text{B}$), and therefore through the three feet of perpendiculars of A, B, C.

If we put A = B = C, so that the $\Delta$ becomes equilateral, the locus reduces to $\quad r = \tfrac{1}{2}a\cos 3\theta.$

We can easily verify this by geometry.

For, ABC being equilateral, all its singular points coincide in I, and as before $\text{IX}' = \text{ED} = \text{EI}\cos\text{DEI}$
$$= \tfrac{1}{2}a\cos(\text{EIA}+\text{EVI})$$
$$= \tfrac{1}{2}a\cos 3\theta.$$




Expressed in rectangular coordinates, the general locus is the sextic
$$(x^2+y^2)^{\frac{3}{2}} = x\,(x\cos\text{C}+y\sin\text{C})$$
$$\times\,(x\cos\text{B}-y\sin\text{B}).$$

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**10123.** (J. Griffiths, M.A.)—If

$\text{I}_3(x, y) = a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3, \quad \text{I}_2(x, y) = a_0x^2+2a_1xy+a_2y^2,$

prove that $x = a_1a_2-a_0a_3$ and $y = 2(a_0a_2-a_1^2)$ will invariantise both $\text{I}_3$ and $\text{I}_2$, and that these quantics will have a common factor, viz., the discriminant of $\text{I}_3$.

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*Solution by* W. Gross, Dr.Sc.

Let $\text{I}_2'$, $\text{I}_3'$ be the transformed values of $\text{I}_2$ and $\text{I}_3$, and R the discriminant of $\text{I}_3$. Then we have

$$\text{I}_2' = a_0\text{R}, \quad \text{I}_3' = (a_0^2a_3-3a_0a_1a_2+2a_1^3)\,\text{R}, \quad a_0^2a_3-3a_0a_1a_2+2a_1^3,$$

being the first coefficient in the cubic covariant Q of $\text{I}_3$.

For *symbolic* calculation, we put $I_3 = a_x^3 = b_x^3 \dots = g_x^3$, therefore

$$I_2 = g_x^2 g_1, \quad g_x = g_1 x + g_2 y = (ab)^2 (bg)\, a_1 \text{ (after the substitution)},$$

$$I_2' = (ab)^2 (bg)\, a_1 (cd)^2 (dg)\, c_1 g_1, \quad I_3' = (ab)^2 (bg)\, a_1 (cd)^2 (dg)\, c_1 (ef)^2 (fg)\, e_1.$$

In order to bring $I_2'$ and $I_3'$ in an invariant form, we introduce $z_1 = 1$, $z_2 = 0$, and find after a short transformation

$$I_2' = (ab)^2 (cd)^2 (bg)\,(dg)\, a_z c_z g_z = -\tfrac{1}{3} (ab)^2 (cd)^2 (ac)\,(bd)\, g_z^3 = \mathrm{R}I_3,$$

$$I_3' = (ab)^2 (cd)^2 (ef)^2 (bg)\,(dg)\,(fg)\, a_z c_z e_z$$
$$= -\tfrac{1}{3} (ab)^2 (cd)^2 (ac)\,(bd)\,(ef)^2 (fg)\, e_z g_z^2 = \mathrm{R}Q_z^3.$$

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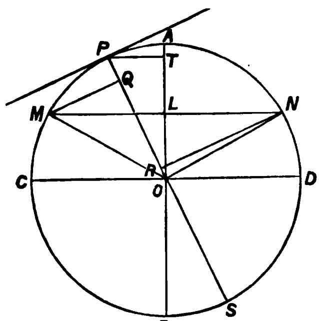
**3216.** (Artemas Martin, LL.D.)—A sphere is cut by a random plane, and then cut again; prove that the chance that the last section is a complete circle is ·3831497 or $\frac{23}{60}$ nearly.

*Solution by* D. Biddle.




The two planes are severally parallel to tangential planes. It is indifferent with which point on the surface of the sphere the tangential plane parallel to the first is in contact; but let A be that point, and AB the diameter perpendicular to the first plane. Let ACBD be the great circle passing through A and P (the similar point in reference to the second of the two planes). Then, MN being the line of intersection of the first plane with ACBD, and MQ, NR perpendiculars upon the diameter PS, it is evident that, in fulfilling the conditions, the second plane may cut PS anywhere in PQ or RS, and the particular chance is (PQ + RS)/PS. For the purposes of the investigation, it is sufficient to suppose P to lie on the quarter-sphere which is bisected by the quadrant ACO. Let AOM = $\phi$, and AOP = $\theta$. Then (PQ + RS)/PS = $1 - \sin\phi \,.\, \sin\theta$. Now, since L, the mid-point of MN, may with equal probability lie anywhere on OA, let OL = $x$; then $\sin\phi = (1-x^2)^{\frac{1}{2}}$, and (PQ + RS)/PS = $1-(1-x^2)^{\frac{1}{2}}\sin\theta$. Again, the points on the quarter-sphere, similar to P, form the circumference of a circle of radius PT, and are as $\sin\theta$. Therefore we have the integral

$$\int_0^1 \int_0^{\frac{1}{2}\pi} \left\{\sin\theta - (1-x^2)^{\frac{1}{2}} \sin^2\theta\right\} dx\, d\theta = \int_0^1 \left\{1-(1-x^2)^{\frac{1}{2}} \,.\, \tfrac{1}{4}\pi\right\} dx$$
$$= 1 - \tfrac{1}{16}\pi^2 = ·3831497, \text{ or } \tfrac{23}{60} \text{ nearly}.$$

**10145.** (Professor Mannheim.)—On donne un triangle $abc$. On trace une circonférence qui passe par $a$, elle coupe $ab$ en $c'$ et $ac$ en $b'$. On trace une circonférence qui passe par $b$ et $c'$, elle coupe $bc$ en $a'$ et la première circonférence en $i$; les points $i$, $a'$, $c$, $b'$ sont sur une même circonférence. On prend un point arbitraire $o$ sur le plan $abc$. La droite $oa$ coupe en $\alpha$ la circonférence qui passe par $a$. La droite $ob$ coupe en $\beta$ la circonférence qui passe par $b$. Enfin sur la troisième circonférence on a le point $\gamma$ à sa rencontre avec $oc$. Démontrer que les points $o$, $\alpha$, $\beta$, $\gamma$, sont sur une même circonférence de cercle.

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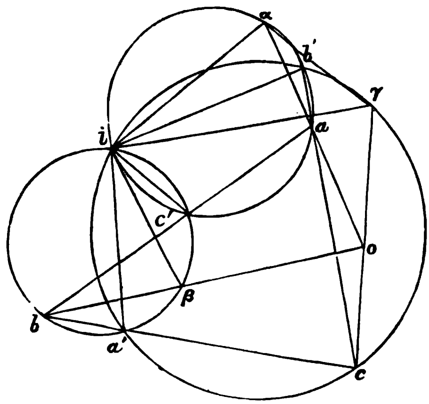
*Solution by* W. J. Greenstreet, M.A.; Professor Darboux; *and others.*

$$\angle b'ic' = \pi - \angle b'ac', \text{ and } \angle a'ic' = \angle c'ba';$$

therefore $\angle b'ia' = \pi - \angle b'ac' + \angle c'ba' = \angle bac + \angle c'ba' = \pi - \angle acb$,

therefore $b'$, $i$, $a'$, $c$ are concyclic.




Again, $\angle i\gamma c = \angle ib'c = \pi - \angle ic'a = \angle ic'b = \angle i\beta b = \pi - \angle i\beta o$,

therefore $i$, $\gamma$, $o$, $\beta$ are concyclic, and

$\angle i\gamma o = \angle ib'a = \angle i\alpha a = \pi - \angle i\beta o$; therefore $i$, $\gamma$, $o$, $\alpha$ are concyclic;

*i.e.*, $i$, $\gamma$, $o$, $\alpha$, $\beta$ are concyclic.

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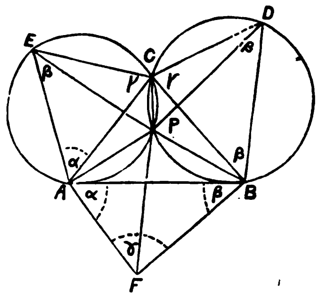
**10158.** (C. H. Thompson, B.A.)—On the sides of a triangle ABC are described three similar triangles BCD, CAE, ABF of any given species, so that the pairs of angles at A, B, C are equal. Prove that the circles described about the three triangles pass through a common point, and that AD, BE, CF meet in this same point.

**10162.** (E. M. Langley, M.A.)—BKC, CLA, AMB are equilateral

triangles described externally on the sides BC, CA, AB of a triangle ABC; prove that AK, BL, CM are equal, and have a common point.

*Solution by* Rev. J. L. Kitchin, M.A.; D. T. Griffiths; *and others.*




(10158.) About CBD, AEC describe circles cutting in P. Join points as in figure.

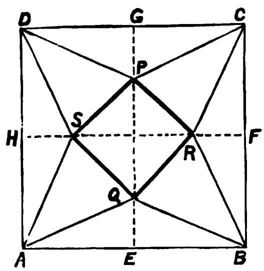
$\angle APC = \alpha+\gamma$, $\angle BPC = \beta+\gamma$, whence $\angle APB = \alpha+\beta$; therefore, in quadrilateral APBF, angles APB, AFB are two right angles, therefore circle about AFB passes through P.

Again, $\angle CPB = \beta+\gamma$, and $\angle EPC = \alpha$, and their sum 2 right angles, therefore EP, PB are in one straight line; so AP, PD are in one straight line, and CP, PF.

(10162.) This is a simple case of this more general proposition [and both are included in Quest. 7818, Vol. XLIII., page 88].

**10166.** (C. A. Swift.)—Equilateral triangles are described on the four sides of a square, the triangles all lying within the square. Show that the area of the eight-pointed star-shaped figure formed by the vertices of the triangle and the corners of the square, together with three times the area of the square, is equal to eight times the area of one of the equilateral triangles.

*Solution by the* Rev. J. L. Kitchin, M.A.; *and* W. J. Greenstreet, M.A.




Let ABCD be a square, its side $2a$. P, R, Q, S the vertices of the four equilateral triangles. There are three star-shaped figures, each suiting the wording of the question; it appears, however, that star, with square PRQS cut out of it, is the one intended by the question.

Now $\quad EP = a\sqrt{3}$,

therefore $\quad EQ = 2a-a\sqrt{3}$,

therefore the four triangles

$$AQB, BRC, \&c. = 4a^2(2-\sqrt{3}).$$

But $\quad PQ = EP-EQ = 2a\sqrt{3}-2a$,

therefore area of square PRQS $= \frac{1}{2}(2a\sqrt{3}-2a)^2 = 8a^2-4a^2\sqrt{3}$,

therefore $\quad$ star $= 4a^2-8a^2+4a^2\sqrt{3}-8a^2+4a^2\sqrt{3}$,

therefore star + 3 times square $= 8a^2\sqrt{3} = 8$ times equilateral triangle.

**10102.** (L. J. ROGERS.)—If $q < 1$, show that the value the infinite of continued fraction $\frac{q(1+q^2)}{1-q^3+}\ \frac{q^2(1+q)(1+q^3)}{1-q^5+}\ \frac{q^3(1+q^2)(1+q^4)}{1-q^7+}$ is $q$.

*Solution by* Professor SEBASTIAN SIRCOM.

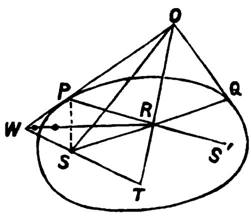
The first three convergents are $q\frac{1+q^2}{1-q^3}$, $q\frac{1-q^5}{1+q^6}$, $q\frac{1+q^9}{1-q^{10}}$, whence the $n^{\text{th}}$ convergent is $q\frac{1+(-1)^{n-1}q^{n(n+3)/2}}{1-(-1)^{n-1}q^{(n+1)(n+2)/2}}$, which is $q$ when $n=\infty$, $q<1$. This is easily proved by induction, observing that the numerator of the $(n-2)^{\text{th}}$ convergent written in full will be

$$q(1+q^2)\dots(1+q^{n-2})\left[1+(-1)^{n-1}q^{(n-2)(n+1)/2}\right],$$

and that the $n^{\text{th}}$ element is $\frac{q^n(1+q^{n-1})(1+q^{n+1})}{1-q^{2n-1}}$.

**10132.** (W. W. POOLE HUGHES, B.A.)—OP and OQ are tangents to an ellipse, whose foci are S and S′. SQ and S′P intersect in R. The normal at P intersects OR produced in T; TS meets the tangent at P in W. Prove that TP, OS, and RW are concurrent.

*Solution by* T. W. ROBINSON, B.A.; G. E. CRAWFORD, B.A.; *and others.*

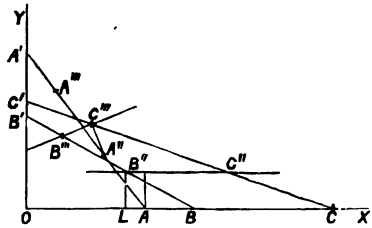



We know that SO and PT bisect angles at S and P of $\Delta$PRS, and PO is perpendicular to PT, therefore O is S-escribed centre of $\Delta$PRS, therefore OR bisects angles PRQ or SRS′, therefore T is P-escribed centre of $\Delta$PRS, and therefore W is R-escribed centre, therefore RW, PT, and OS are concurrent.

**10042.** (Professor NEUBERG.)—On donne deux droites rectangulaires OX, OY. Soient A, B, C trois points quelconques de OX; A′, B′, C′ trois points quelconques de OY; A″, B″, C″ trois points divisant les droites AA′, BB′, CC′ dans le rapport $\alpha : 1$; A‴, B‴, C‴ trois points divisant ces droites dans le rapport $\beta : 1$. Cela posé: démontrer que (1) les perpendiculaires abaissées de A sur B″C″, de B sur C″A″, de C sur A″B″ concourent en un même point P; (2) lorsque $\alpha$ varie, P décrit une droite; (3) les perpendiculaires abaissées de A″ sur B‴C‴, de B″ sur C‴A‴, de C″ sur A‴B‴ concourent en un point Q; (4) lorsque $\alpha$ varie, Q décrit une droite; (5) lorsque $\beta$ varie, Q décrit une hyperbole équilatère.

*Solution by* L. WIENER, LL.D.

Let $OA = a_1$, $OB = a_2$, $OC = a_3$;
$OA' = b_1$, $OB' = b_2$, $OC' = b_3$.




(1) The coordinates of B″ are B″L, OL, but

$$\frac{B''L}{OB'} = \frac{B''B}{B'B} = \frac{\alpha}{\alpha+1},$$

$$B''L = \frac{\alpha}{\alpha+1} \cdot b_2,$$

$$\frac{OL}{BO} = \frac{B'B''}{B'B} = \frac{1}{\alpha+1}, \quad OL = \frac{1}{\alpha+1} \cdot a_2.$$

Similarly the coordinates of C″ are

$$\left(\frac{1}{\alpha+1} \cdot a_3, \quad \frac{\alpha}{\alpha+1} \cdot b_3\right), \text{ and of A'' are } \left(\frac{\alpha}{\alpha+1} \cdot a_1, \quad \frac{\alpha}{\alpha+1} \cdot b_1\right).$$

The equation of the line B″C″ is

$$\left\{(\alpha+1)\,X - a_2\right\} / (a_2 - a_3) = \left\{(1-\alpha^{-1})\,y - b_2\right\} / (b_2 - b_3).$$

Hence that of the perpendicular line through A is

$$\alpha^{-1}(a_2 - a_3)(x - a_1) + (b_2 - b_3)\,y = 0.$$

Similarly, the equation of the perpendicular line to A″C″ through B is

$$\alpha^{-1}(a_3 - a_1)(x - a_2) + (b_3 - b_1)\,y = 0,$$

and of the perpendicular line to A″B″ through C is

$$\alpha^{-1}(a_1 - a_2)(x - a_3) + (b_1 - b_2)\,y = 0.$$

Adding, the equation vanishes, therefore the three perpendicular lines pass through a point.

(2) Eliminating $\alpha$ from the first and second equation, we get

$$\frac{(a_2 - a_3)(x - a_1)}{b_2 - b_3} = \frac{(a_3 - a_1)(x - a_2)}{b_3 - b_1},$$

the equation of a straight line and parallel to the axis of $y$.

(3) The coordinates of A‴, B‴, C‴ are

$$\frac{1}{\beta+1} \cdot a_1, \quad \frac{\beta}{\beta+1} \cdot b_1; \quad \frac{1}{\beta+1} \cdot a_2, \quad \frac{\beta}{\beta+1} \cdot b_2; \quad \frac{1}{\beta+1} \cdot a_3, \quad \frac{\beta}{\beta+1} \cdot b_3.$$

The equation of B‴C‴ is

$$\left\{(\beta+1)\,x - a_2\right\} / (a_2 - a_3) = \left\{(1+\beta^{-1})\,y - b_2\right\} / (b_2 - b_3),$$

and that of the perpendicular line to B‴C‴ through A″ is

$$(a_2 - a_3)\left\{(\alpha+1)\,x - a_1\right\} + \beta\,(b_2 - b_3)\left\{(\alpha+1)\,y - \alpha b_1\right\} = 0.$$

The other equations are symmetrical, and vanish identically when added; hence the three perpendiculars pass through a point.

(4) Eliminating $\alpha$ between two equations, we get

$$\frac{(a_2 - a_1)(x - a_3) + \beta\,(b_2 - b_1)\,y}{(a_2 - a_1)\,x + \beta\,(b_2 - b_1)(y - b_3)} = \frac{(a_3 - a_2)(x - a_1) + \beta\,(b_3 - b_2)\,y}{(a_3 - a_2)\,x + \beta\,(b_3 - b_2)(y - b_1)},$$

or $$\frac{\beta(b_2-b_1)\,b_3-(a_2-a_1)\,a_3}{(a_2-a_1)\,x+\beta(b_2-b_1)(y-b_3)}=\frac{\beta(b_3-b_2)\,b_1-(a_3-a_2)\,a_1}{(a_3-a_2)\,x+\beta(b_3-b_1)(y-b_1)},$$
which is evidently the equation of a straight line.

5. Eliminating $\beta$ between two equations, we have
$$\frac{(a_2-a_3)\left\{(\alpha+1)\,x-a_1\right\}}{(b_2-b_3)\left\{(\alpha+1)\,y-\alpha b_1\right\}}=\frac{(a_2-a_1)\left\{(\alpha+1)\,x-a_3\right\}}{(b_2-b_1)\left\{(\alpha+1)\,x-\alpha b_3\right\}},$$
which is of the form $Axy+Bx+Cy+D=0$; transferring the equation to a new coordinate system, parallel to the former, we can make the coefficients of $x$ and $y$ vanish, leaving the equation $Axy+D'=0$, which is the equation of a hyperbola referred to its asymptotes; but these make a right angle with each other, hence the hyperbola is equilateral.

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**9650.** (Fannie H. Jackson, B.Sc.)—Prove that (1) the circles that circumscribe the four triangles got by omitting successively each of four lines pass through a point; and (2) their centres lie on a circle that passes through the same point.

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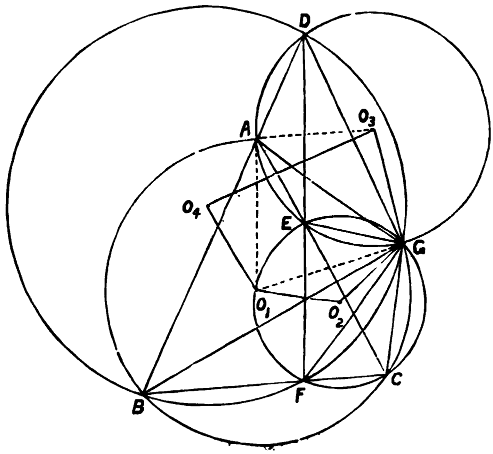
*Solution by* W. J. Greenstreet, M.A.; *and* Rev. J. J. Milne, M.A.

AB, AC, BC, DF make triangles ABC, EFC, DAE, BDF.

1. Let the circles round ADE, EFC cut in G. Then
$$\angle AGC = \angle AGE + \angle EGC = \pi - \angle EFC + \angle BDF$$
$$= \angle BFD + \angle BDF = \pi - DBF,$$



therefore the circumcircle of ABC passes through G. Also

$$\angle \text{FGC} = \angle \text{FEC} = \angle \text{DEA} = \text{DGA},$$

therefore $\angle \text{AGC} = \angle \text{DGF}$, and also $\angle \text{AGC} = \pi - \angle \text{DBF}$

therefore $\angle \text{DGF} = \pi - \angle \text{DBF}$,

therefore the circumcircle of DBF passes through G.

2. Again, drawing all common chords, we have, since they are perpendicular to joins of centres respectively,

$$\angle O_3O_4O_1 = \pi - \angle \text{DGB} = \angle \text{DBG} + \angle \text{BDG} = \tfrac{1}{2}(\text{AO}_1\text{G} + \text{AO}_3\text{G})$$
$$= \tfrac{1}{2}\pi - \text{O}_1\text{AG} + \tfrac{1}{2}\pi - \text{O}_3\text{AG} = \pi - \text{O}_1\text{AO}_3 = \pi - \angle \text{O}_1\text{GO}_3$$

(the triangles $AO_1O_3$, $GO_1O_3$ are clearly equal in all respects),

therefore $O_1$, $O_4$, $O_3$, G are concyclic. Also

$$\left.\begin{array}{l} O_4O_1O_2 = \pi - \angle \text{BGC} = \angle \text{DAC} \\ O_4O_3O_2 = \angle \text{DGE} = \pi - \angle \text{DAC} \end{array}\right\}, \quad \text{therefore } O_4O_1O_2 + O_4O_3O_2 = \pi,$$

or $O_2$ lies on the circumcircle of $O_1O_4O_3$. Hence the angular points of the pentagon $O_1O_2GO_3O_4$, are concyclic.

[See *Solutions* to MILNE's *Weekly Problem Papers*, p. 232, No. 5; also McDOWELL's *Exercises*, 3rd edit., No. 119.]

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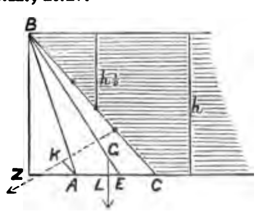
**10066.** (Professor ASÛTOSH MUKHOPÂDHYÂY, M.A.)—An embankment of triangular section ABC supports the pressure of water on the side BC; find (1) the condition of its not being overturned about the angle A when the water reaches to B, the vertex of the triangle; and prove (2) that, if $s$ is the specific gravity of the embankment, when the area of the triangle is reduced to the minimum consistent with stability for a given depth of water,

$$\tan A = (s^2 + 2s + 9)^{\frac{1}{2}} / (s - 1), \quad \tan C = (s^2 + 2s + 9)^{\frac{1}{2}} / (3 - s).$$

*Solution by* C. MORGAN, M.A., R.N.




Considering a section of the embankment, the resultant pressure of the water on BC acts two-thirds of the way down. Let $h$ = depth of water. Hence, taking moments about A, we must have, for the condition of the embankment not being overturned,

$$g \cdot 1 \cdot \tfrac{1}{2}h \cdot \text{BC} \cdot \text{AK} \not< \tfrac{1}{2}h \cdot \text{AC} \cdot s \cdot g \cdot \text{AL},$$

$$\text{BC} \cdot \text{AK} \not< \text{AC} \cdot \text{AL} \cdot \text{S},$$

$$\text{BC}\,(\tfrac{1}{3}\text{BC} - \text{AC}\cos C) \not< \text{AC}\,(\tfrac{2}{3}\text{AC} - \tfrac{1}{3}\text{BC}\cos C)\, s;$$

$$\therefore \quad \tfrac{1}{3}a^2 - ab\cos C \not< (\tfrac{2}{3}b^2 - \tfrac{1}{3}ab\cos C)\, s \quad \text{or} \quad \frac{3c^2 - 3b^2 - a^2}{3b^2 - a^2 + c^2} \not< s.$$

Let $AZ = k$, then $\dfrac{3h^2 + 3k^2 - 3b^2 - h^2 - k^2 - 2bk - b^2}{3b^2 - h^2 - k^2 - b^2 - 2bk + h^2 + k^2} = s,$

when the embankment is just turning over,

or $$\frac{h^2+k^2-2b^2-bk}{b^2-bk} = s, \quad \text{when } b \text{ is a minimum } \frac{db}{dk} = 0.$$

Therefore $$-bs = 2k-b \quad \text{or} \quad \frac{b}{k} = \frac{2}{1-s},$$

$$s = \frac{(h^2/k^2)(1-s)^2+(1-s)^2-8-2(1-s)}{2+2s},$$

$$2s+2s^2 = (h^2/k^2)(1-s)^2+s^2-9, \quad s^2+2s+9 \equiv \tan^2 A \,.\, (1-s)^2,$$

$$\therefore \quad \tan A \equiv \pm \frac{(s^2+2s+9)^{\frac{1}{2}}}{1-s}, \quad \tan C = \frac{1-s}{3-s}\tan A = \frac{(s^2+2s+9)}{3-s}.$$

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**9644.** (R. Knowles, B.A.)—The circle of curvature is drawn at a point P of a conic; M is the mid-point of the common chord; O the centre of curvature; the diameter of the conic through M meets the normal at P in Q; prove that $OQ : MO = e^2 : 2-e^2$, $e$ being the eccentricity.

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*Solution by* Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

Let $(\theta, -3\theta)$ be the eccentric angles of P and the other end of the common chord; then the coordinates of M and CM are

$$(a\cos 2\theta\cos\theta, \;\; -b\cos 2\theta\sin\theta), \quad a\cos\theta \,.\, y + b\sin\theta \,.\, x = 0 \;\;......(1).$$

Q is the point of intersection of (1) with the normal at P, that is, with

$$a\sin\theta \,.\, x - b\cos\theta \,.\, y = c^2\sin\theta\cos\theta\,;$$

therefore Q is $\left\{c^2a\cos\theta\,/\,(a^2+b^2), \;\; -c^2b\sin\theta\,/\,(a^2+b^2)\right\}$,

and O is $\left\{c^2\cos^3\theta\,/\,a, \;\; -c^2\sin^3\theta\,/\,b\right\}$.

Hence $$OQ = c^2(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}(a^2\sin^2\theta-b^2\cos^2\theta)\,/\,\left\{ab\,(a^2+b^2)\right\},$$

$$OM = (a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}(a^2\sin^2\theta-b^2\cos^2\theta\,/\,(ab).$$

Thus $$OQ : OM = c^2 : a^2+b^2 = a^2-b^2 : a^2+b^2 = e^2 : 2-e^2.$$

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**8395.** (By Professor Clarke.)—Given four points on a plane no three of which are collinear, prove that (1) there is one and *only one* ellipse of minimum area passing through the four points; (2) the solution of the problem depends on a cubic equation, and discuss the roots of this cubic; (3) the congruent root of this cubic makes the ellipse a minimum and not a maximum; note (4) the case when the four points are the angular points of a parallelogram; and hence (5) deduce *immediately* a solution of the problem of describing the minimum ellipse through *three* given points.

*Solution by* D. EDWARDES.

(1) Using trilinear coordinates, let $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ be the equation of the ellipse, S its area, and $\alpha_1, -\beta_1, \gamma_1$ the coordinates of the fourth point. Then, putting $\frac{al}{cn} = x$, $\frac{bm}{cn} = y$, we have

$$S = \frac{4\pi\Delta xy}{\{2xy + 2y + 2x - x^2 - y^2 - 1\}^{\frac{3}{2}}}$$
$$= \frac{4\pi\Delta B^2 x (C + Ax)}{\{2x(B^2 + p^2) - x^2(A-B)^2 - (B-C)^2\}^{\frac{3}{2}}} \quad\text{......(1)},$$

the first value being subject to the condition $Ax - By + C = 0$, where

$$A = \frac{1}{a\alpha}, \quad B = \frac{1}{b\beta}, \quad C = \frac{1}{c\gamma},$$

thus giving the second value, where $p^2 \equiv B(A+C) - CA$ and is evidently positive. Since the denominator must be real, we easily find that $x$ must lie between $\left(\frac{B+p}{A-B}\right)^2$ and $\left(\frac{B-p}{A-B}\right)^2$. For a real solution, therefore, $x$ must be positive. Differentiating (1), we have for a maximum or minimum

$$A(A-B)^2x^3 + \{2C(A-B)^2 + A(B^2+p^2)\}x^2$$
$$-\{2A(B-C)^2 + C(B^2+p^2)\}x - C(B-C)^2 = 0,$$

say $px^3 + qx^2 - rx - s = 0$. Since $p, q, r, s$ are evidently all positive, we see that this equation has only one positive root, and therefore there can be *only one* solution.

(2) We have seen that the solution depends upon a cubic having only one positive root. Now, writing the cubic in the form

$$(C + 2Ax)\{2x(B^2+p^2) - x^2(A-B^2) - (B-C)^2\}$$
$$-3x(C+Ax)\{B^2 + p^2 - x(A-B)^2\} = 0$$

substituting for $x$ $\left(\frac{B+p}{A-B}\right)^2$, ($x_1$ say), and observing that $(A-B)(B-C$ $= p^2 - B^2$, the left-hand member reduces to $6x_1(C + Ax_1)pB$; and, substituting for $x$, $x_0$ or $\left(\frac{B-p}{A-B}\right)^2$, the left-hand member becomes $-6x_0(C + Ax_0)pB$. Hence a root of the cubic lies between $x_1$ and $x$ which must therefore be its positive root.

(3) Let $f(x) = A(A-B)^2x^3 + \{\quad\}x^2 - \{\quad\}x - C(B-C)^2$; and let $\alpha$ be the positive root of $f'x = 0$. Then $f'x = A(A-B)^2(x-\alpha)R$, where R is positive for all positive values of $x$. Hence, as $x$ increases through $\alpha$, $f'x$ changes from negative to positive. And $f''x$ cannot *vanish* when $x = \alpha$, for then $f'x = 0$ would have two positive roots, which is impossible.

(4) In this case we must have $A = B = C$, $\therefore x = 1$, $y = 2$, and $\therefore al = cn = \frac{1}{2}bm$. If $\alpha_0, \beta_0, \gamma_0$ are the coordinates of the centre,

$$\frac{\alpha_0}{l(bm + cn - al)} = \frac{\beta_0}{m(cn + al - bm)} = \frac{\gamma_0}{n(al + bm - cn)}.$$

Hence $\alpha_0 = \frac{\Delta}{a}$, $\beta_0 = 0$, $\gamma_0 = \frac{\Delta}{c}$, which verifies the preceding work. The minimum area in this case is to the area of the parallelogram as $\pi : 2$.

(5) Here $\frac{ds}{dx} = 0$, $\frac{ds}{dy} = 0$. These lead to $x = y = 1$, and the centre of the ellipse is at the centroid of the triangle. I do not see how to deduce this case immediately, but perhaps the above solution may be of some interest.

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**10148.** (Professor WOLSTENHOLME, Sc.D.)—Prove that, if

$$(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^n(\theta x),$$

$$\int_0^\infty \frac{f^n(\theta x)}{x^r}\,dx = \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)}\int_0^\infty \frac{f^n(x)}{x^r}\,dx,$$

being any positive quantity. [If $r > 1$, both integrals generally $= \infty$.]

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*Solution by* E. B. ELLIOTT, M.A.

Professor WOLSTENHOLME has given a solution of this question in the *Proceedings of the London Mathematical Society*, Vol. XIII., p. 185. Again, I have myself considered the question in the *Messenger of Mathematics*, Vol. XII., p. 144, where I have shown that the equality

$$\int_0^\infty \frac{f^n(\theta x)}{x^r}\,dx$$

$$= \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)}\int_0^\infty \frac{f^n(x)}{x^r}\,dx + \Gamma(n+1)\frac{d}{dr}\left\{\frac{\Gamma(r)}{\Gamma(n+r)}\right\}\left[\frac{f^n(x)}{x^{r-1}}\right]_0^\infty$$

one which is identical with Professor Wolstenholme's for all cases in which his is intelligible as an equivalence of finite quantities; for, if the first member on the right is finite, it is essential that the second is zero, and which, when written in the form

$$\int_0^\infty \left\{f^n(\theta x) - \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)}f^n(x)\right\}\frac{dx}{x^r} = \Gamma(n+1)\frac{d}{dr}\left\{\frac{\Gamma(r)}{\Gamma(n+r)}\right\}\left[\frac{f^n(x)}{x^{r-1}}\right]_0^\infty,$$

s true, and intelligible as an equivalence of finite quantities, in another and wider class of cases; viz., when the difference of limits $\left[\frac{f^n(x)}{x^{r-1}}\right]_0^\infty$ is finite. The only case in which the theorem must be regarded as only indicated rather than proved, is that in which the value of $\frac{f^n(x)}{x^{r-1}}$ when $x = \infty$ or $x = 0$, though not infinite, is not a definite limit $\left(\text{as is, for instance, the case if } \frac{f^n(x)}{x^{r-1}} = \cos x \text{ or } \sin x\right)$. Precisely the same uncertainty, if it be uncertainty, is then attached to the conclusion as attaches to the value of such an integral as $\int_0^\infty \frac{\sin kx}{x}\,dx$.

**2842.** (MORGAN JENKINS, M.A.)—In Degen's table of the quotients to be used to form the convergents to the value of $\sqrt{N}$, where N is any non-square integral number from 1 to 1000, it is seen that the number of quotients in the period (excluding the first quotient which does not recur) never exceeds $2a$, when N lies between $a^2$ and $(a+1)^2$. Can this be proved generally?

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*Solution by* ARTEMAS MARTIN, LL.D.

It is *not* true generally. Let $a = 90$, then 8269 is between $a^2$ and $(a+1)^2$. The number of terms in a period of $\sqrt{(8269)}$ is 181, which exceeds $2a = 180$.

Let $a = 94$, then 8941 is between $a^2$ and $(a+1)^2$. The number of terms in a period of $\sqrt{(8941)}$ is 207, which exceeds $2a = 188$.

Let $a = 99$, then 9949 is between $a^2$ and $(a+1)^2$. The number of terms in a period of $\sqrt{(9949)}$ is 217, which exceeds $2a = 198$.

The number of terms in a period of the square roots of these numbers were first found by Mr. C. A. ROBERTS, of Ohio, who is preparing a table of the cycles for primes of the form $4n+1$ for publication in the *Mathematical Magazine*, and afterwards verified by Mr. SYLVESTER ROBINS, of New Jersey.

The square root of any non-quadrate number $A^2+b$ can be expressed as a periodic continued fraction of the form

$$A+\frac{1}{d_1+}\frac{1}{d_2+}\frac{1}{d_3+}\dots\frac{1}{d_n+}\frac{1}{d_n+}\dots+\frac{1}{d_3+}\frac{1}{d_2+}\frac{1}{d_1+}\frac{1}{2A+}\dots$$

when $n$ is even, and of the form

$$A+\frac{1}{d_1+}\frac{1}{d_2+}\frac{1}{d_3+}\dots+\frac{1}{d_n+}\frac{1}{d_{n+1}+}\frac{1}{d_n+}\dots+\frac{1}{d_3+}\frac{1}{d_2+}\frac{1}{d_1+}\frac{1}{2A+}\dots$$

when $n$ is odd. What is the *greatest* value $n$ can have?

A complete general solution of this problem is a *desideratum* in the Theory of Continued Fractions.

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**2560.** (J. J. WALKER, F.R.S.)—Given that either of one pair of impossible roots of the equation $3x^4-16x^3+30x^2+8x+39=0$ gives a real result when substituted for $x$ in $5x^3-18x^2-7x$, it is required to find the four (impossible) roots of the biquadratic.

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*Solution by the* Rev. J. L. KITCHIN, M.A.

Assume as one pair $\alpha \pm i\beta$, then $5(\alpha+i\beta)^3-18(\alpha+i\beta)^2-7(\alpha+i\beta)$ is real; hence $\qquad \alpha^2-\frac{12}{5}\alpha = \frac{1}{15}(5\beta^2+7) = \frac{9}{5}$ if $\beta = 2$.

Thus $\alpha = 3$, $\beta = 2$, and one trial pair being $3 \pm i \,.\, 2$, the quadratic factor is $x^2-6x+13=0$. The other factor is found to be $3x^2+2x+3=0$; and roots $-\frac{1}{3}\pm\frac{2}{3}\sqrt{2}\,i$, the four roots are $3\pm 2i$, $-\frac{1}{3}+\frac{2}{3}\sqrt{2}\,i$.

**10086.** (W. J. GREENSTREET, M.A.)—**Through any point M on the normal to an ellipse at P, draw normals MQ, MR, MS. Find the locus of the ortho-, the circum-, and mass centres of the triangle QRS.**

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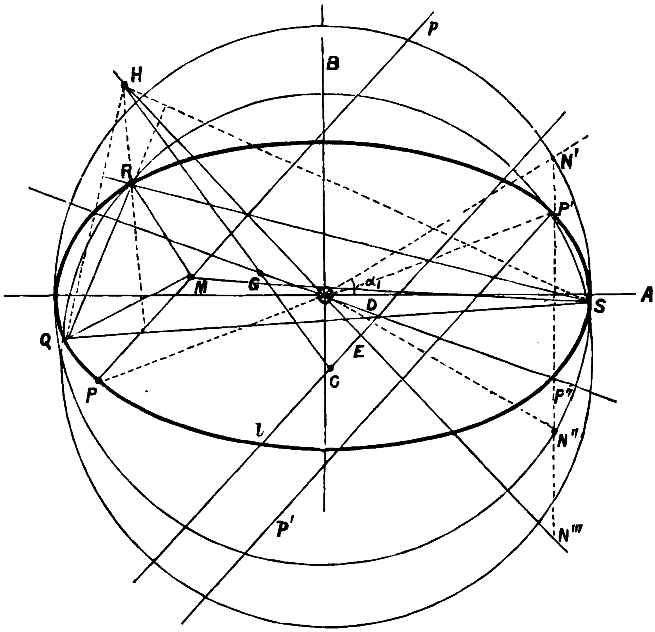
*Solution by* Professor SCHOUTE.

In the diagram P is the given point, P′ the point diametrically opposite to P, and $l$ the line parallel to the normals $p$ and $p'$ in P and P′ that bisects the distance of the centre O and $p'$. According to the known theorems of JOACHIMSTAHL the circle through P′, the centre of which is




any point C of $l$, meets the ellipse in three other points Q, R, S, the normals of which concur in a point M of $p$. So the locus of the circum-centre C is the line $l$, that is complementary to the normal $p$ with reference to O (expression due to E. HAIN).

The locus of the centre of gravity G and that of the orthocentre H are two diameters. The first OP″ is antiparallel to OP with reference to the axes. And the second ON‴ is antiparallel to the diameter conjugated to OP with reference to the bisectors of the angle AOB formed by the axes.

The second and third parts are demonstrated analytically in the following way. Let $\alpha_1$ be the excentrical anomaly of P′, then the equation of $l$ is $2ax \sin \alpha_1 - 2by \cos \alpha_1 = c^2 \sin \alpha_1 \cos \alpha_1$. So we find for the coordinates

of C the values $x_c = (k^2+c^2)\cos\alpha_1/4a$ and $y_c = (k^2-c^2)\sin\alpha_1/4b$, $k^2$ being an arbitrary constant. Thus the circle QRS is

$$\left(x-\frac{k^2+c^2}{4a}\cos\alpha_1\right)^2+\left(y-\frac{k^2-c^2}{4b}\sin\alpha_1\right)^2$$
$$=\left(a-\frac{k^2+c^2}{4a}\right)^2\cos^2\alpha_1+\left(b-\frac{k^2-c^2}{4b}\right)^2\sin^2\alpha_1,$$

and the condition that the point, the eccentric anomaly of which is $\alpha_1$, lies on this circle is

$$2a^2\cos^2\alpha+2b^2\sin^2\alpha=(k^2+c^2)\cos\alpha\cos\alpha_1-(k^2-c^2)\sin\alpha\sin\alpha_1=a^2+b^2-k^2.$$

This equation leads to the coordinates of G. If we put $a\cos\alpha = u$, $a\cos\alpha_1 = u_1$, we have $c^4u^4-c^2(k^2+c^2)u^3u_1+\ldots=0$. So we find

$$u_1+3x_g=(k^2+c^2)u_1/c^2,\quad i.e.,\ x_g=ak^2\cos\alpha_1/3c^2.$$

If we put $b\sin\alpha = v$, $b\sin\alpha_1 = v_1$, we find in the same manner

$$y_g=-bk^2\sin\alpha_1/3c^2.$$

And by elimination of $k^2$ the locus of G proves to be the line $y = -bx\tan_g\alpha_1/a$.

Since HG = 2GC, we have $z_h+2z_c=3z_g$, where $z$ stands for $x$ or $y$. So we get $x_h = \mathrm{K}\cos\alpha_1/a$, $y_h=-\mathrm{K}\sin\alpha_1/b$, where

$$\mathrm{K}=\left\{(a^2+b^2)k^2-c^4\right\}/2c^2;$$

by elimination of K the locus of H is found to be the line $y=-ax\tan_g\alpha_1/b$.

To this solution we add the following considerations.

1. The Euler-line CGH of the triangle QRS envelopes a parabola, when M describes the normal $p$. For it meets the three loci $l$, OP″, ON‴ and the line at infinity $l_\infty$ in four points with an invariable biquotient. This parabola touches the sides DE, OD, OE of the triangle DEO formed by the three loci in points O′, E′, D′ respectively for which 2O′D = DE, OE′ = 2E′D, D′O = 2OE. Any other tangent $t$ of this parabola is the locus of a point T of the line of Euler, that forms with C, G, H four points with invariable biquotient. So the locus of the centre of the nine-points circle (*i.e.*, the mid-point of CH) also describes a line, the line through the mid-points of the segments O′E, OD, D′E.

The equation of the parabola enveloped by the line of Euler is

$$\left\{ax(3a^2+b^2)\sin\alpha+by(a^2+3b^2)\cos\alpha\right\}^2-2ac^4x(3a^2+5b^2)\sin^2\alpha\cos\alpha$$
$$-2bc^4y(5a^2+3b^2)\sin\alpha\cos^2\alpha+c^8\cos^2\alpha\sin^2\alpha=0.$$

2. When P describes the given ellipse, the line $l$ envelopes the evolute of another ellipse, coaxial with the given one and with axes of half the dimensions. In that case the two other loci OD and OE envelope the centre O. And the envelope of the locus $t$ of the point T, for which CT $= p$TG, or

$$2(1+p)\left\{ax(3c^2-4b^2p)\sin\alpha_1-by(3c^2+4a^2p)\cos\alpha_1\right\}$$
$$=(3+2p)c^4\cos\alpha_1\sin\alpha_1,$$

is easily found by the remark, that when A = B,

$$\mathrm{A}x\sin\alpha_1+\mathrm{B}y\cos\alpha_1=\mathrm{C}\cos\alpha_1\sin\alpha_1,$$

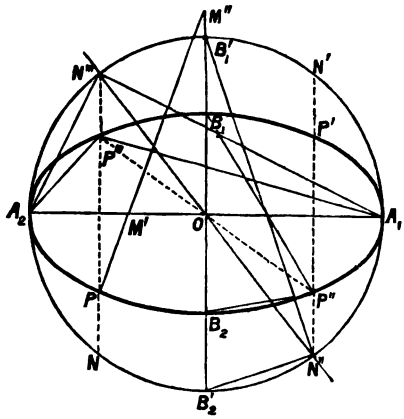
envelops the evolute of the ellipse, the axes $a_1$, $b_1$ of which coincide with the coordinate axes, and are given in length by the relations

$$a_1 C = (a_1^2 - b_1^2)\, A, \quad b_1 C = (a_1^2 - b_1^2)\, B.$$

So we have here
$$(3 + 2p)\, c^4 a_1 = 2\,(1 + p)(3c^2 - 4b^2 p)\, a c_1^2,$$
$$(3 + 2p)\, c^4 b_1 = 2\,(1 + p)(3c^2 + 4a^2 p)\, b c_1^2,$$

or
$$a_1 = \frac{(3 + 2p)(3c^2 - 4b^2 p)\, ac^2}{2\,(1 + p)(9c^4 - 48a^2b^2p - 16a^2b^2p^2)},$$
$$b_1 = \frac{(3 + 2p)(3c^2 + 4a^2 p)\, bc^2}{2\,(1 + p)(9c^4 - 48a^2b^2p - 16a^2b^2p^2)}.$$

3. If M describes the normal $p$, the points Q, R, S form an involution on the given ellipse. The so-called "involutions curve" (*Journal von Crelle*, t. LXXII., p. 287), *i.e.*, the envelope of the sides RS, SQ, QR, is a parabola. To prove this, we firstly remark that the envelope is a conic; for through any point Q of the given ellipse pass two tangents QR, QS to the envelope. In the second diagram we determine this conic by means of the two triangles $A_1A_2P'''$ and $B_1B_2P''$, that correspond to the positions M′ and M″ of M on $p$. These triangles are the projections of the

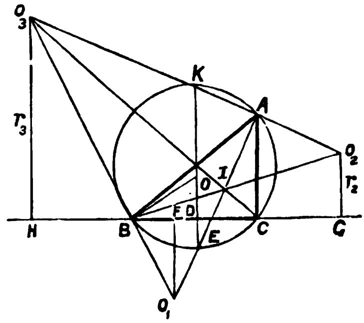



rectangular triangles $A_1A_2N'''$ and $B_1'B_2'N''$, when the ellipse is considered as the projection of the circle. But the conic touching the sides of these triangles is a parabola, of which N″N‴ is the directrix, for the tangents through N″, O, N‴ to this curve are perpendicular to one another. And it is evident that N′ is the focus of this parabola. The projection of this parabola, *i.e.*, the involutions curve, is a parabola, the axis of which is parallel to the tangents in P″ and P‴ at the ellipse.

4. The obtained results are particular cases of more general ones, that correspond to oblique normals (compare *Wiskundige opgaven*, t. II., p. 264, question 160, and *Mathesis*, t. VII., p. 38).

**10009.** (Rev. W. T. Wellacott, M.A.)—Prove, geometrically, that the sum of the perpendiculars on the sides of a triangle from its circum-centre is equal to the sum of the radii of the in-circle and circum-circle.

*Solution by* W. J. Greenstreet, M.A.; E. Rutter; *and others.*




Take O, $O_1$, $O_2$, $O_3$, I, the circum-, ex-, in-centres.

Then $\quad HD = DG,$

$\therefore \quad 2DK = r_2 + r_3.$

$IO_1$ is bisected in E,

$\therefore \quad 2DE = r_1 - r,$

$\therefore \; 2EK = r_1 + r_2 + r_3 - r = 4R.$

$DE = \frac{1}{2}(r_1 - r) = d$, &c.;

$\therefore \quad 2R - r = d + d' + d'';$

and $OD + \ldots + DE + \ldots = 2R,$

therefore $\qquad OD + \ldots = 3R - (2R - r) = R + r.$

**3276.** (Artemas Martin, LL.D.)—Give all the different square numbers that can be made with the nine digits, using all the digits once (and only once) in each number.

*Solution by* D. Biddle.

There are 29 such square numbers, and they are as follows :—

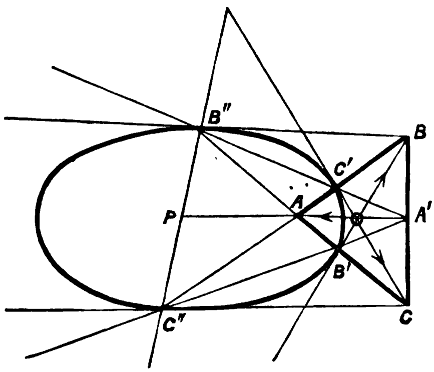
| | |
|---|---|
| 1. $139854276 = 11826^2$. | 16. $587432169 = 24237^2$. |
| 2. $152843769 = 12363^2$. | 17. $589324176 = 24276^2$. |
| 3. $157326849 = 12543^2$. | 18. $597362481 = 24441^2$. |
| 4. $215384976 = 14676^2$. | 19. $615387249 = 24807^2$. |
| 5. $245893761 = 15681^2$. | 20. $627953481 = 25059^2$. |
| 6. $254817369 = 15963^2$. | 21. $653927184 = 25572^2$. |
| 7. $326597184 = 18072^2$. | 22. $672935481 = 25941^2$. |
| 8. $361874529 = 19023^2$. | 23. $697435281 = 26409^2$. |
| 9. $375468129 = 19377^2$. | 24. $714653289 = 26733^2$. |
| 10. $385297641 = 19629^2$. | 25. $735982641 = 27129^2$. |
| 11. $412739856 = 20316^2$. | 26. $743816529 = 27273^2$. |
| 12. $523814769 = 22887^2$. | 27. $842973156 = 29034^2$. |
| 13. $529874361 = 23019^2$. | 28. $847159236 = 29106^2$. |
| 14. $537219684 = 23178^2$. | 29. $923187456 = 30384^2$. |
| 15. $549386721 = 23439^2$. | |

**10058.** (Col. H. W. L. Hime.)—ABC is a given triangle, and the sides are divided in A′, B′, C′ so that BA′ : A′C = $n : m$, CB′ : B′A = $l : n$, AC′ : C′B = $m : l$, where $l-m-n = 0$; also B″, C″ are taken in CA, AB, where C′A′, A′B′ meet them; and AA′, BB′, CC′ meet in O. Prove that, if OA is produced to P, so that OP = 3OA, the points B″, C″, and P are collinear; BB″ and CC″ are parallel; and B″C″ is the diameter of a conic to which ABC is self-conjugate; BB″ touching at B″, CC″ at C″, CC′ at C′, and BB′ at B′. Prove that, if $m = n$, P will be the centre of the conic.

*Solution by the* Proposer.

Let three vectors OA ($\alpha$), OB ($\beta$), OC ($\gamma$) be connected together by the relation, $l\alpha+m\beta+n\gamma = 0$, and complete triangle ABC. Then

$$\frac{\mathrm{BA}'}{\mathrm{A}'\mathrm{C}} = \frac{n}{m}; \quad \frac{\mathrm{CB}'}{\mathrm{B}'\mathrm{A}} = \frac{l}{n}; \quad \frac{\mathrm{AC}'}{\mathrm{C}'\mathrm{B}} = \frac{m}{l}.$$




In investigating the curve (in anharmonics) $x^2-y^2-z^2 = 0$, we find that ABC is self-conjugate with respect to it, and that BB″, BB′, CC″, CC′ are tangents at B″, B′, C″, C′, respectively.

If $l-m-n = 0$, BB″ and CC″ are parallel, and the curve is also an ellipse, the criterion for which is $m^2+n^2 < l^2$. The coordinates of P, the intersection of the two known lines, B″C″ and OA, are $(2, -1, -1)$.

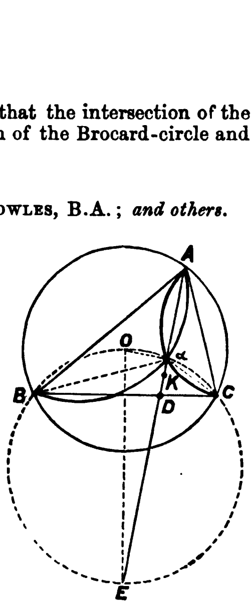
Hence $$\mathrm{OP} = \frac{2l\alpha - m\beta - n\gamma}{2l-m-n} = \frac{2l\alpha + l\alpha}{2l-l} = 3\alpha = 3\mathrm{OA}.$$

Therefore OP = 3OA, or the line OP is trisected in A.

**10018 & 10060.** (10018.) (J. Lemaire.)—Soit O le centre du cercle circonscrit à un triangle ABC. Lieu de la projection du point O sur la symédiane relative au sommet A quand le triangle se déforme de manière que l'angle A demeure constant, les sommets B et C restant fixes.

(10060.) (E. M. LANGLEY, M.A.)—Prove that the intersection of the Brocard-axes $A\Omega_1C$, $A\Omega_2B$ is the intersection of the Brocard-circle and the symmedian through A.

*Solution by* E. M. LANGLEY, M.A.; R. KNOWLES, B.A.; *and others.*




Let the Brocard-arcs $A\Omega_1C$, $A\Omega_2B$ intersect in $\alpha$, and let $A\alpha$ cut BC in D. Therefore AB, AC are tangents at A.

$\angle BA\alpha = \angle AC\alpha$ and $\angle AB\alpha = \angle CA\alpha$

therefore triangles $AB\alpha$, $AC\alpha$ are equiangular.

$\therefore \quad BA^2 : AC^2 :: \triangle AB\alpha : \triangle AC\alpha,$

$:: BD : DC;$

therefore AD is the symmedian through A.

Again, $\angle C\alpha D = CA\alpha + AC\alpha = BA\alpha$.

Similarly $\quad \angle B\alpha D = BA\alpha$.

$\therefore$ (1) $\alpha$ lies on circle through B, C and the circum-centre O of $\triangle ABC$;

and (2) AD passes through the other end E of the diameter OE of this circle.

Therefore, if K be the symmedian point, $\angle O\alpha K$ is a right angle; therefore $\alpha$ lies on the Brocard-circle of $\triangle ABC$.

Now, if (10018) BC remains fixed, and $\angle BA\alpha$ constant, O and the circle BOC remain fixed; therefore the locus of the projection $\alpha$ of O on AD is the fixed arc BOC.

Note that the projections $\alpha, \beta, \gamma$ of the circumcentre on the symmedians lie each at the cointersection of the Brocard-circle and two Brocard-arcs.

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**10125.** (CH. HERMITE.)—Démontrer que, lorsque $m$ est un multiple de $n$, $\{m(m-1)(m-2)\dots(m-n+2)\}/n!$ est un nombre entier.

*Solution by* Professor G. B. MATHEWS, M.A.

The given expression is $m!/\{n!(m-n+1)!\}$, and it has to be shown that any prime factor which occurs in the denominator occurs at least as often in the numerator.

First, let the prime not divide $n$: then it will occur in $n!(m-n+1)!$ just as often as in $(n-1)(m-n+1)!$ and since $m!/\{n-1)!(m-n+1)!\}$ is an integer, the required result follows in this case.

Next, if the prime divides $n$ and therefore $m$, it will not divide $(m-n+1)$, and it will therefore occur just as often in $n!(m-n+1)!$ as in $n!(m-n)!$ Now $m!/\{n!(m-n)!\}$ is an integer; therefore, as before, and the proposition is proved.

**10170.** (R. KNOWLES, B.A.)—The sides AB and CD of a quadrilateral inscribed in a conic meet in E; the diagonals AC and BD intersect in G; prove that the four points E, G, and the poles of AD and CB are collinear.

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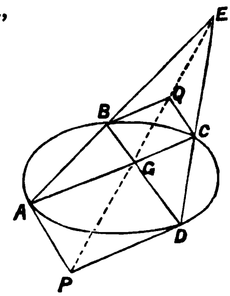
*Solution by* D. T. GRIFFITHS; *and* G. G. STORR, M.A.




$$A\{ABCD\} = D\{ABCD\} = D\{DCBA\};$$

therefore $A\{PEGD\} = D\{PEGA\}$.

Since these two equal pencils contain one common ray AD, the corresponding rays

$$\left.\begin{matrix}AP.\\DP.\end{matrix}\right\}\left.\begin{matrix}AE\\DE\end{matrix}\right\}\left.\begin{matrix}AG\\DG\end{matrix}\right\}$$

intersect on a straight line; hence P, G, E are collinear.

In the same way it can be shown that Q, G, E are collinear.

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**10075.** (Professor NILKANTHA SARKAR, M.A.)—The vertices ABCD of a quadrilateral are to be connected by a system of straight lines; prove that (1) the systems AP, BP, PQ, QC, CD and BP, CP, PQ, QA, QD, where P, Q are such that the angles at P, Q are all equal, are systems whose total lengths are minima; (2) if equilateral triangles AEB, CED, BGC, AHD be drawn outside the quadrilateral, the length of the first system is equal to EF, and of the second GH; and that if $d$, $d'$ are the lengths of the diagonals AC, BD, and $\alpha$ the angle between them, these lengths are the two values of

$$\left\{d^2 + d'^2 \pm 2dd' \cos(\alpha \mp \tfrac{1}{3}\pi)\right\}^{\frac{1}{2}}.$$

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*Solution by* Professor SYAMADAS MUKHOPADHYAY, B.A.

Join EF. Let the circumcircles of AEB, CFD intersect EF at P, Q. Take any two points P′, Q′. Then, since the sum of the distances of any point from two vertices of an equilateral triangle is equal to or greater than the distance from the third vertex as the point lies on the circumcircle or elsewhere, we have

$$AP' + BP' + P'Q' + Q'C + Q'D$$

$$\text{not} \;< EP'P'Q' + Q'F,$$

that is, greater than EF, and

$$EF = EP + PQ + QF$$

$$= AP + BP + PQ + QC + QD.$$

This proves the first and second parts of the question.

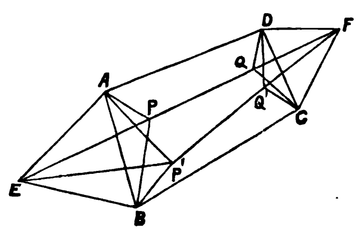


Fig. 1.

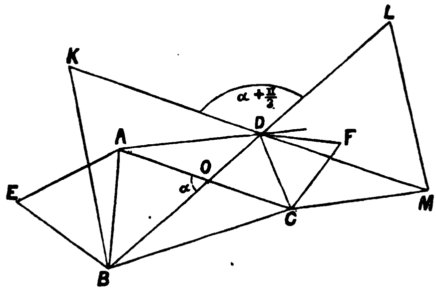


Fig. 2.

Complete the parallelogram ADMC and describe the equilateral triangles DML, BDK as in Fig. 2.

Then EK = AD = CM = FL, also EK is parallel to FL, since EK, FL are inclined at 60° to AD, CM. Hence

$$EF = KL = \left\{d^2 + d'^2 - 2dd' \cos(\alpha + \tfrac{1}{3}\pi)\right\}^{\frac{1}{2}}.$$

This proves the third part.

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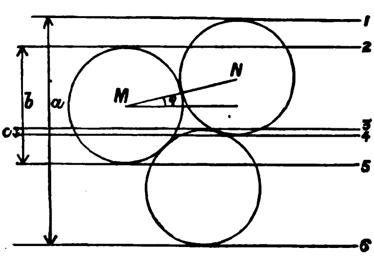
**10251.** (D. BIDDLE.)—A figure, consisting of three equal circles touching each other, is disposed at random on a floor ruled with parallel lines, the distance between which exceeds that of two diameters. Find the probability that the figure will be cut by one of the lines, (1) in two points, (2) in four points, (3) in six points.

*Solution by* Professor P. H. SCHOUTE.




In the diagram $\phi$ denotes the inclination of MN on the parallel lines of the floor; 1, 2, 3, 4, 5, 6 are the tangents to the circles parallel to these lines; and $(a-b)/l$, $(b-c)/l$, $c/l$ indicate the relative probabilities, when the angle $\phi$ is given, and $l$ is the distance of the floor lines. Now, $b = 2r$,

$$a = 2r\left\{1 + \cos(\tfrac{1}{6}\pi - \phi)\right\},$$
$$c = 2r\left\{1 - \cos(\tfrac{1}{6}\pi - \phi)\right\},$$

when $r$ is the radius of the circles; hence the probabilities are

$$W_1 = \frac{2r}{\frac{1}{6}\pi l}\int_0^{\frac{1}{6}\pi} \cos(\tfrac{1}{6}\pi - \phi)\, d\phi = \frac{6r}{\pi l}, \quad W_2 = \frac{6r}{\pi l},$$

$$W_3 = \frac{2r}{\frac{1}{6}\pi l}\int_0^{\frac{1}{6}\pi} \left\{1 - \cos(\tfrac{1}{6}\pi - \phi)\right\} d\phi = \left(2 - \frac{6}{\pi}\right)\frac{r}{l}.$$

**9987.** (Professor DE WACHTER.)—A sphere, acted on by gravity, rolls down a surface of revolution with vertical axis. Find at which point of the generating curve the sphere will leave the surface, supposing the generatrix to be (1) a circle; (2) an ellipse; (3) a cycloid.

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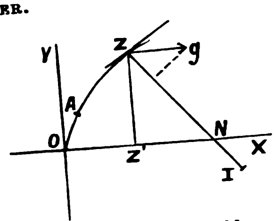
*Solution by the* PROPOSER.




The generating plane curve is referred to a system of rectangular coordinates having its origin at the vertex O, and the positive X-axis being taken along the direction of gravity. The motion is supposed to take place along the generating curve on the outer or convex side of the surface, the rolling sphere being reduced to a material point starting without initial velocity from a given point A ($x_0$, $y_0$).

In consequence of this motion, an increasing amount of centrifugal force is called into play; it is directly opposed by that component of gravity which acts normally to the generatrix. The moving point is kept on the surface by the same component until contact ceases beyond a certain point Z ($x$, $y$) where both contending forces cancel each other, and free motion sets in.

Denoting by $v$ the velocity, by $n$ the normal ZN, and by $\rho$ the radius of curvature ZI, we have

$$\frac{v^2}{\rho} = g\,\frac{(n^2-y^2)^{\frac{1}{2}}}{n}.$$

But
$$v^2 = 2g\,(x-x_0).$$

Hence
$$2n\,(x-x_0) = \rho\,(n^2-y^2)^{\frac{1}{2}} \quad \text{.............................(1)},$$

Introducing the expressions of $n$ and $\rho$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, account being taken of their relative signs, we get

$$2\,(x-x_0)\frac{d^2y}{dx^2} + \frac{dy^3}{dx^3} + \frac{dy}{dx} = 0 \quad \text{.......................(2)}$$

The coordinates of Z may now be determined for each curve, by substituting the corresponding values of $n$ and $\rho$ in (1), or those of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (2).

(A) Thus, if the generatrix be an *ellipse*, with axes $2a$, $2b$ $(a \gtrless b)$, we arrive at

$$x^3 - 3ax^2 - 3\,\frac{a^2b^2}{a^2-b^2}\,x + \frac{2ax_0+b^2}{a^2-b^2}\,a^3 = 0 \quad \text{......................(3)}.$$

Putting $x = z+a$, $x_0 = z_0+a$, we have

$$z^3 - 3z\,\frac{a^4}{a^2-b^2} + 2\,\frac{a^4z_0}{a^2-b^2} = 0 \quad \text{..........................(4)}.$$

This equation admits three real roots if $a^2 > b^2$; *i.e.*, if gravity acts along the major axis. In this case the question is answered by the smallest root ($z$ and $x$). It admits but one real root if $a^2 < b^2$; *i.e.*, if gravity acts along the minor axis.

(B). If $a = b$, the ellipse becomes a *circle*, and the equation (3) reduces to $3x = 2x_0 + a$, or $x - x_0 = \frac{1}{3}(a - x_0)$.

(C). The generatrix is a *cycloid* (*a*) with *vertical base*. By applying (2) we get $x = x_0 + r \sin \omega$ ($r$ = radius of the generating circle). But $x = r\omega - r \sin \omega$. Hence $\omega$ is given by $r(\omega - 2 \sin \omega) = x_0$.

(*b*) With *horizontal base*. Since the radius of curvature = twice the corresponding normal as measured between the curve and the base, we get easily $2(x - x_0) = 2r - x_0$, *i.e.*, the vertical height of A above the base is twice that of Z.

[The relation (2) expresses the condition required in order that the normal component of gravity may be, at any point of the generatrix, equal and opposite to the centrifugal force exerted at the same point. It is therefore the differential equation to the generatrix of the revolution-surface which undergoes no pressure from the rolling or sliding particle. Hence the required curve must be a parabola. In fact, by integrating (2), we obtain $(y - c)^2 = 4p(x - p)$, which represents a parabola referred to a rectangular system formed by its horizontal directrix and any vertical axis. The required motion may therefore be described as follows:—The particle starts from any point in the directrix and falls through a vertical distance until it meets the curve in T where its direction is suddenly changed along that of the tangent in T, its velocity remaining unaltered. The motion continues freely along the parabola.]

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**9960 & 10121.** (E. Lemoine.)—On circonscrit à toutes les ellipses homofocales de foyers F et F′ des rectangles dont la direction des côtés est donnée; démontrer que tous les points de contact appartiennent à une même hyperbole équilatère qui passe par F et F′ et a pour asymptotes les parallèles menées par le centre des ellipses aux côtés des rectangles.

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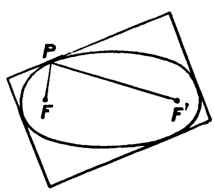
*Solution by* Professor Schoute; G. E. Crawford, B.A.; *and others.*

Comme les droites FP et F′P sont toujours antiparallèles par rapport aux directions des côtés, le lieu de P est une hyperbole équilatère concentrique, qui passe par F et F′ et dont les asymptotes ont ces directions données. (See Vol. li., p. 66.)

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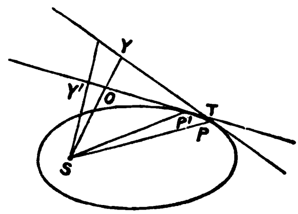
**10005.** (R. Lachlan, M.A.)—If SY be the perpendicular from the focus S of an ellipse on the tangent at the point P, find the position of P when the area of the triangle SPY is a maximum.

*Solution by* H. W. SEGAR; C. BICKERDIKE; *and others.*




Let PY, P′Y′ be consecutive tangents; SY, S′Y′ the perpendiculars on them, meeting in O. Then evidently, for a maximum, we have

$$\Delta OSY' = \Delta OPP' + \Delta TOY$$

ultimately; that is,

$$SY^2 = SY \, . \, \rho + PY^2,$$

$\rho$ being the radius of curvature at P.

Let $SY = p$, $SP = r$, then this becomes
$$r\frac{dr}{dp} = \frac{2p^2 - r^2}{p} = 2p - \frac{r^2}{p}.$$

Now, $l$ being the semi-latus rectum, the equation of the ellipse is

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a}, \text{ or } p^2 = \frac{alr}{2a-r},$$

therefore
$$\frac{2dp}{p} = \left\{\frac{1}{r} + \frac{1}{2a-r}\right\} dr = \frac{2a}{r(2a-r)} dr.$$

Substituting in $\dfrac{pdr}{dp} = \dfrac{2p^2}{r} - r$, we get

$$\frac{r(2a-r)}{a} = \frac{2al}{2a-r} - r,$$

therefore $r(2a-r)^2 = 2a^2l - ar(2a-r)$, or $ar^3 - 5a^2r^2 + 6a^3r - 2a^2b^2 = 0$, an equation for the distance of P from S.

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**10112.** (Professor MOREL.)—On considère deux cercles concentriques O. On trace un troisième cercle variable C, tangent en un point fixe A du plus petit des cercles O et coupant le plus grand aux points M, N; on joint M et N au point B diamétralement opposé à A dans le cercle C. Démontrer que les droites BM et BN enveloppent un ellipse fixe.

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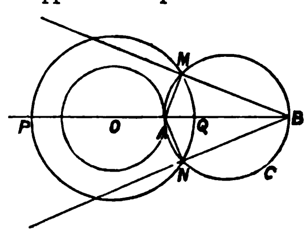
*Solution by* R. KNOWLES, B.A.; J. C. ST. CLAIR; *and others.*




It is immediately evident that the lines BM and BN envelop the ellipse with the major axis PQ of which A is a focus.

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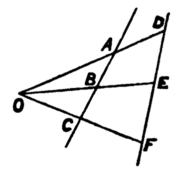
**10126.** (H. W. SEGAR.)—The straight lines ABC, DEF cut three others which meet in a point O in the points A, B, C, D, E, F; prove that

$$AD \, . \, BC/OD + BE \, . \, CA/OE + CF \, . \, AB/OF = 0.$$

*Solution by* Professors SCHOUTE, BEYENS, *and others.*




$$\Delta EOF + \Delta FOD + \Delta DOE = 0, \text{ or } \Sigma \Delta EOF.$$

This gives immediately in a transparent notation the identities

$$\Sigma \frac{\sin EOF}{OD} = 0, \quad \Sigma \frac{OA}{OD} . OB . OC \sin BOC = 0,$$

or $$\Sigma \frac{OA}{OD} \Delta BOC = 0, \quad \Sigma \frac{OA . BC}{OD} = 0.$$

By subtraction of $\Sigma BC = 0$, we get the identity in question.

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**9525.** (F. R. J. HERVEY.)—Prove that, through any two sets of mutually orthocentric points, each formed by the intersection of two pairs of perpendicular tangents to a three-cusped hypocycloid, a rectangular hyperbola can be drawn, cutting the cycloid generally at the ends of two or four complete tangent chords, which are respectively parallel to the normals of the hyperbola at its intersections with the circle through the vertices of the cycloid; the tangents to the hyperbola at such intersections being, as well as the asymptotes, tangents to the cycloid.

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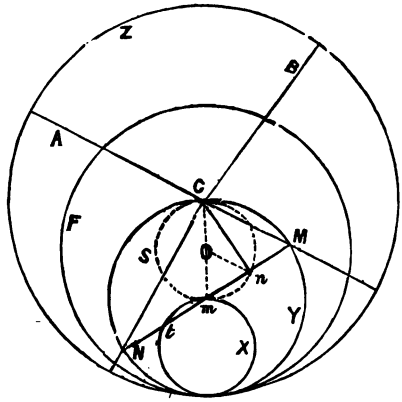
*Solution by the* PROPOSER.




The circles X, Y, Z rolling upon F (centre O), the diameters A, B (at right angles) of Z and MN of Y envelope, and points $t$ on X, and M, N on Y describe, the same three-cusped hypocycloid with vertices on S. When, as in the figure, X, Y, Z touch F at a common point, $t$ is the point of contact of MN, and M, N those of A, B. C$n$ is the third tangent through C. We are concerned with the system of hyperbolas having the revolving line-pair AB for asymptotes. The mid-point $m$ of the tangent chord MN, of constant length, may be called briefly the mid-point of the tangent.

We have the following ratios of angular velocities:

Vel. O$m$ : vel. MN $= -2$, vel. OC : vel. AB ($=$ vel. O$n$ : vel. MN) $= 4$. It follows that, whatever the relative positions of AB and any third

tangent MN, the directions $Cm$, MN are always conjugate with respect to AB; or, the mid-point of the tangent is also that of its intercept by any hyperbola of the system. Hence, every set of four points, the intersections of two line-pairs, has S for its nine-point circle; every hyperbola of the system which passes through one point of such a set passes through all; and through any two sets one hyperbola can be drawn.

One point of a set may be taken arbitrarily within the cycloid. If taken on the cycloid or on S, the set reduces to three points such as C, M, N. Thus, to each intersection C of a hyperbola with S, correspond two, M, N, with the cycloid; and $Cn$ is tangent at C (its mid-point) to all hyperbolas passing through that point.

Any tangent (of cycloid) is a Simson line to the triangles of any set (whence, the asymptotes of a rectangular hyperbola are Simson lines of every inscribed triangle). For, its intercepts by the three line-pairs having a common mid-point on S, reflection of the whole figure about that point shows that the six perpendiculars meet by threes at four points, the reflections of those of the set.

Hyperbolas meeting without the cycloid have only two real intersections. The analogy which exists between this system of hyperbolas, with their sets of common intersections, and that of lines and points in the plane, has partly appeared, and may be completely exhibited in the manner briefly indicated below. Numberless propositions can then be deduced.

Let V be a vertex, V′ opposite cusp, V′OV axis of $x$, $OC = 1$, $2\phi$ the angle VOC or V′O$m$, $\xi$, $\eta$ coordinates of any point $p$, and P the set of points obtained by solving the equations

$$y^2-x^2-2x = \xi, \quad 2xy-2y = \eta \qquad (1).$$

The equation
$$\xi \sin\phi + \eta\cos\phi + \sin 3\phi = \mu \qquad (2)$$
is that of a line which becomes the tangent MN when $\mu = 0$; by substitution from (1), it becomes that of a hyperbola which reduces to its asymptotes A, B when $\mu = 0$ ($\mu$ being, apart from sign, the squared semi-axis). The *line* (2) is the polar of O with respect to the *hypberbola* (2), reflected with respect to O. The point $p$ is the orthocentre of the pedal triangle of the orthocentric set P (when real), similarly reflected [verified by observing that, from (1), we have

$$2\left\{(-\xi/2-x)^2+(-\eta/2-y)^2\right\}^{\frac{1}{2}} = x^2+y^2.]$$

Substitution from (1) converts a locus of $p$ into the corresponding locus of P. It is noteworthy that, if $p$ describe the trochoid

$$\xi = 2c\cos\theta - c^2\cos 2\theta, \quad \eta = 2c\sin\theta + c^2\sin 2\theta,$$

one point of P describes the circle $x^2+y^2 = c^2$.

For three line-pairs, centres $a$, $b$, $c$, meeting in a set,

$$\angle\, \mathrm{VO}a + \mathrm{VO}b + \mathrm{VO}c = 0.$$

For three tangents meeting at a point, read *mid-point* for *centre*, and $\mathrm{V}_1$ for V. For a tangent triangle with S for mid-circle and tangents for Simson lines, restore V.

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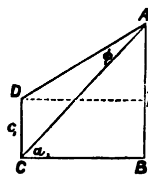
**10062.** (Professor Aiyar, M.A.)—A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye, and

that the depressions of their bases are $a_1, a_2, a_3$; prove that, $c_1, c_2, c_3$ being the heights of the towers, $\frac{\sin(a_2-a_3)}{c_1 \sin a_1} + \frac{\sin(a_3-a_1)}{c_2 \sin a_2} + \frac{\sin(a_1-a_2)}{c_3 \sin a_3} = 0.$

---

*Solution by* C. MORGAN, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

$$\frac{\sin\phi}{\cos a_1} = \frac{c_1}{\mathrm{AD}} = \frac{c_1}{(h-c_1)\sec(90^\circ+\phi-a_1)}$$
$$= \frac{c_1}{(h-c_1)\cos(a_1-\phi)},$$




therefore
$$\left.\begin{aligned} h-c_1 &= c_1\cos a_1(\sin a_1\cot\phi-\cos a_1) \\ h-c_2 &= c_2\cos a_2(\sin a_2\cot\phi-\cos a_2) \\ h-c_3 &= c_3\cos a_3(\sin a_3\cot\phi-\cos a_3) \end{aligned}\right\}$$

Eliminating $\cot\phi$,

$$h(c_2\sin a_2\cos a_2 - c_1\sin a_1\cos a_1) = c_1c_2\sin a_1\sin a_2\sin(a_1-a_2),$$
$$h(c_1\sin a_1\cos a_1 - c_3\sin a_3\cos a_3) = c_1c_3\sin a_1\sin a_3\sin(a_3-a_1),$$
$$h(c_3\sin a_3\cos a_3 - c_2\sin a_2\cos a_2) = c_2c_3\sin a_2\sin a_3\sin(a_2-a_3);$$

hence by addition we have the required result.

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**9963.** (R. KNOWLES, B.A.)—A circle touches a conic in a point P, and cuts it again in Q, R; M, N are the points of contact on the conic of the two real common tangents meeting in T; prove that (1) the lines MN, QR, and the tangent at P are concurrent; (2) if K be the pole of QR with respect to the conic, the points P, T, K are collinear.

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*Solution by* Rev. T. GALLIERS, M.A.; G. G. STORR, M.A.; *and others.*

This Question is a particular case of the following more general theorem:—"If two conics are inscribed in the same triangle ABC, touching BC in the same point P, then (1) the polars of the two conics with respect to A and their common chord are concurrent, their point of intersection lying on BC; (2) the pole of the common chord with respect to either conic lies on the line AP."

To prove this, let the equations of the two conics be

$$S \equiv u^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 - 2vw\beta\gamma - 2wu\gamma\alpha - 2uv\alpha\beta = 0,$$
$$S' \equiv u'^2\alpha^2 + v^2\beta^2 + w^2\gamma^2 - 2vw\beta\gamma - 2wu'\gamma\alpha - 2u'v\alpha\beta = 0.$$

Then (1) the equations of the polars of these conics with respect to A are respectively $u\alpha - v\beta - w\gamma = 0$, $u'\alpha - v\beta - w\gamma = 0$;

also $\quad S - S' \equiv (u-u')\,\alpha\,\{(u+u')\,\alpha - 2v\beta - 2w\gamma\} = 0,$

so that the equation of the common chord of the conics is

$$(u+u')\,\alpha - 2v\beta - 2w\gamma = 0 \quad \text{..........................(1).}$$

Hence we see that the two polars and the common chord meet BC in the point given by $\alpha = 0$, $v\beta + w\gamma = 0$.

(2) The equation of AP is clearly $v\beta - w\gamma = 0$.

If $(f, g, h)$, $(f', g', h')$ be the poles of (1) with respect to the conics S, S′ respectively,

$$(v^2 . g - 2vw . h - 2uv . f) / (-2v) = (w^2 . h - 2vw . g - 2wu . f)(-2w),$$

$$(v^2 . g' - 2vw . h' - 2u'v . f') / (-2v) = (w^2 . h' - 2vw . g' - 2wu' . f')(-2w),$$

therefore $vg - wh = 0$ and $vg' - wh' = 0$; thus both poles lie on AP.

Question 9963 is what the above proposition becomes when one of the conics is a circle.

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**10000.** (J. W. Russell, M.A.)—Prove the following rule for the power of the modulus in the case of any covariant or invariant of any number of quantics in any number of variables, viz.:—Consider each variable except one of dimensions 0 in length, and consider the other variable to be of $-1$ dimensions, and take the dimensions of each coefficient to be such that each term in the quantic is of 0 dimensions, then the power of the modulus is the dimensions of the covariant or invariant or, briefly, the power of the modulus of any covariant or invariant is the *reduced* dimensions of the covariant or invariant.

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*Solution by the* Proposer.

For clearness consider first the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Here $x$ and $y$ are of 1 dimensions in length. The corresponding quantic is $ax^2 + 2hxy + by^2 + 2gzx + 2fyz + cz^2$, where $z$ is of 0 dimensions in length. The dimensions (or weights) of the coefficients, and the reduced dimensions of the variables, are

$$\begin{array}{cccccccccc} a, & b, & c, & f, & g, & h, & \quad & x, & y, & z, \\ 0, & 0, & 2, & 1, & 1, & 0, & \quad & 0, & 0, & -1. \end{array}$$

As an example take $\quad abc + 2fgh - af^2 - bg^2 - ch^2$.

This being the Hessian, we know that the power of the modulus is 2. Also the reduced dimensions of the terms are

$$abc + 2fgh - af^2 - bg^2 - ch^2,$$

$$(0+0+2) + (1+1+0) - (0+2) - (0+2) - (2+0),$$

*i.e.*, in each case 2.

To prove the rule, make use of the transformation

$$x = \lambda x', \quad y = y', \quad z = z', \ \ldots\ldots,$$

where $x$ is the unit-variable (of lower dimensions than the others). This gives $\Delta = \lambda$, and the proof presents no difficulty.

As another example (see Salmon's *Solid Geometry*, § 218) consider the developable generated by the tangent lines of the curve common to two quadrics U and V. One term is $\Phi^2 U^2 V^2$.

The reduced dimensions of U and V are by hypothesis 0.

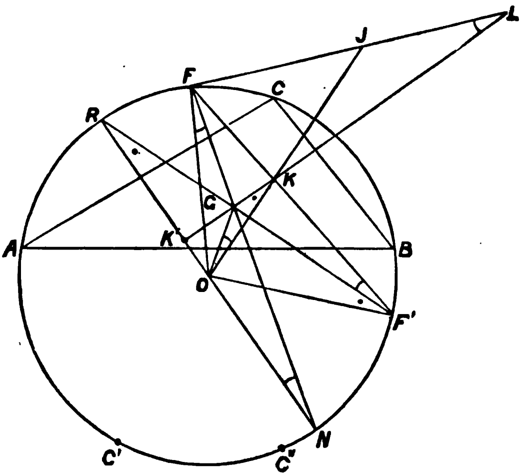
Again, a term in $\Phi$ is $bca'd'$, which is of dimensions $(0+0+0+2)$ if we take $w$ as unit-variable.

Hence the reduced dimensions of the covariant is 4, and therefore 4 is the required power of the modulus.

**10239.** (E. VIGARIÉ.)—Soient ABC un triangle et G son centre de gravité. Les droites AG, BG, CG coupent le cercle circonscrit respectivement en A′, B′, C′. Démontrer que le point de LEMOINE du triangle A′B′C′ est sur le diamètre qui passe par les points de TARRY et de STEINER du triangle ABC.

*Solution by* W. S. McCAY, M.A.

Let the circumcircle be inverted into itself, at G as origin; the three circles through G and the extremities C, C″; A, A″; B, B″ of the symmedian chords of the circle invert into the symmedian chords of the new triangle A′B′C′, and L, the intersection of the three circles, inverts




into K′ the new symmedian point, for these circles cut the circumcircle in C″, &c., the harmonic conjugate of C with respect to AB, &c.; hence the inverse corresponding chords cut the circle in the same manner with reference to the triangle A′B′C.

The point L may be constructed without drawing the symmedian chords.

Let J be the inverse of K with respect to the circumcircle; then the circle OGJ cuts GK in L, for KG . KL = KO . KJ, the power of K with respect to the circumcircle. I will show that L inverts from G into K′ on the diameter NOR.

A property of this line is that the triangles RNG, KOG are similar. This is known, and results from G being the double point of ABC and M. Brocard's triangle $A_1B_1C_1$. NG bisects OK.

Let NG, RG cut the circle in F, F′. The four points O, G, K, F′ are seen to be on a circle; hence F, K, F′ are in one line; therefore the points F, J, L are in one line, the inverse with respect to K, of this circle. The

circle FGL is the inverse with respect to G of the line NO, for it goes through F the inverse of N, and cuts the circumcircle orthogonally there. Hence L inverts into K′ on the diameter NOR.

The trilinear coordinates of the point F′ are $a/(b^2+c^2-2a^2)$, &c. This point is the focus of a parobola inscribed in ABC, and whose axis is parallel to GK. I have not seen *this* point noticed. We see it is intimately connected with the diameter NOR.

The coordinates of F are known to be cosec (B−C), &c. F is the focus of a parabola inscribed in ABC, whose directrix is OG.

[See Vol. XLV., p. 115, and Prof. NEUBERG's "Point de Steiner."]

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**9165.** (Professor BORDAGE.)—If a triangle having a constant angle is deformed in such a manner that, the summit of the constant angle being fixed and the opposite side passing through a fixed point, one of the two other summits describes a straight line, prove that the third summit describes *a conic*.

*Solution by* C. TAYLOR, D.D.

Let A be the fixed summit, B that which describes a straight line, C the third summit, D the fixed point through which BC passes. Then, by cross ratio, $$A\{C\} = A\{B\} = D\{C\},$$
and the locus of C is a conic through A and D.

Or, by making AB and AC successively coincide with AD, show that A and D are single points on the locus. Then, since there is one other position only of C on a line through A or D, the locus is of the second order.

[The theorem is the special case of Newton's organic description of conics and other curves used in Professor CAYLEY's Solution of Question 1409, Vol. I., p. 40 in 1st ed. (1864), or 78 in 2nd ed. (1886). For another solution of Question 9165, see Vol. L., p. 107. In the general case, BDC is any constant angle turning about D.]

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**10036.** (Professor HUDSON, M.A.)—Two rough inclined planes make equal angles $\alpha$ with the horizon, and are separated by a short hard smooth horizontal plane: a perfectly elastic particle is projected directly up one of the inclined planes with a velocity just sufficient to carry it to a distance $a$, it then descends and after impact ascends the other plane: find the height to which it will rise; also the whole time of motion before it is brought to rest and the aggregate distance traversed in that time, the angle of friction being $\lambda$.

*Solution by* Rev. T. R. TERRY, M.A.

Let $v_n$ be the velocity with which the particle starts up the $n^{\text{th}}$ plane, and $a_n$ the distance it gets up that plane; then, since the acceleration in

going up is $g\sin(\alpha+\lambda)\sec\lambda$, and in coming down is $g\sin(\alpha-\lambda)\sec\lambda$, the formula $v^2 = 2fs$ gives

$v_{n+1}^2 : v_n^2 = \sin(\alpha-\lambda) : \sin(\alpha+\lambda)$, $\therefore\ a_{n+1} : a_n = \sin(\alpha-\lambda) : \sin(\alpha+\lambda)$; therefore the whole space traversed is

$$2a\sin(\alpha+\lambda)\left\{\sin(\alpha+\lambda)-\sin(\alpha-\lambda)\right\}^{-1} = \frac{a\sin(\alpha+\lambda)}{\cos\alpha\sin\lambda}.$$

Again, for brevity, putting $p^2 \equiv \sin(\alpha+\lambda)$ and $q^2 \equiv \sin(\alpha-\lambda)$, the formula $s = \frac{1}{2}ft^2$ gives

$$\text{Time of making all ascents} = \frac{(2ag^{-1}\cos\lambda)^{\frac{1}{2}}}{p}\cdot\frac{p}{p-q},$$

$$\text{Time of making all descents} = \frac{(2ag^{-1}\cos\lambda)^{\frac{1}{2}}}{q}\cdot\frac{p}{p-q},$$

therefore $$\text{Total time} = \frac{(2ag^{-1}\cos\lambda)^{\frac{1}{2}}}{q}\cdot\frac{p+q}{p-q}.$$

[This solution assumes that $\alpha > \lambda$, but if $\alpha \not> \lambda$, distance $= a$, and time $= \left(\dfrac{2a\cos\lambda}{g\sin(\alpha+\lambda)}\right)^{\frac{1}{2}}$.]

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**10136.** (C. A. Swift, B.A.)—Prove that the expansion of tan tan tan ... tan $x$, the tangent being taken $n$ times, is

$$x + 2n\frac{x^3}{3!} + 4n(5n-1)\frac{x^5}{5!} + \frac{8n}{3}(175n^2-84n+11)\frac{x^7}{7!} + \dots$$

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*Solution by* Rev. T. R. Terry, M.A.

Let $f(n)$ be the required expansion; then, since

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots,$$

we may clearly put $f(n) = x + a_n\dfrac{x^3}{3!} + b_n\dfrac{x^5}{5!} + c_n\dfrac{x^7}{7!}$,

where $a_0 = b_0 = c_0 = 0$. Equating coefficients in the identity

$$f(n-1) = \tan^{-1}f(n),$$

we get $$\frac{a_{n-1}}{3!} = \frac{a_n}{3!} - \frac{1}{3},\quad \text{therefore } a_n = 2n;$$

$$\frac{b_{n-1}}{5!} = \frac{b_n}{5!} - \frac{1}{3}\left(\frac{3a_n}{3!}\right) + \frac{1}{5},\quad \text{therefore } b_n = 4n(5n-1);$$

$$\frac{c_{n-1}}{7!} = \frac{c_n}{7!} - \frac{1}{3}\left(\frac{3b_n}{5!} + 3\frac{a_n^2}{3!\,3!}\right) + \frac{1}{5}\left(\frac{5a_n}{3!}\right) - \frac{1}{7},$$

therefore $$c_n = \frac{8n}{3}\left\{175n^2-84n+11\right\}.$$

**10180.** (Professor SYLVESTER, F.R.S.)—$n$ round beads of equal size are arranged in a circular necklace each in contact with its neighbour on either side. $i$ of the beads it is required shall admit of being cut by the same right line. Find the least value of $n$ that will serve to fulfil this condition.

*Solution by* JOHN J. BARNIVILLE.

Inscribe in the circle a regular $n$-gon. First let $i$ be even. Draw a tangent at $A_1$, and on it a perpendicular $A_{\frac{1}{2}i}P$. Then, since the radius of each bead is $r\sin\pi/n$, it is required that $A_{\frac{1}{2}i}P \not< 2r\sin\pi/n$, that is,

$$\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+\ldots+\sin\frac{i-1}{n}\pi \not< 1.$$

The minimum value of $n$ is therefore given by the equation

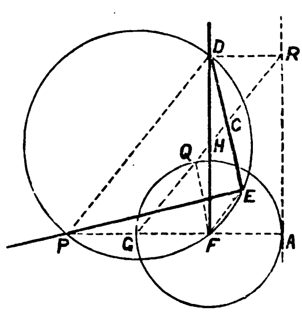
$$\sin\frac{i\pi}{2n}\sin\frac{i\pi}{n}-\sin\frac{\pi}{n}=0.$$

If $i$ is odd, substitute $i+1$ for $i$.

**10051.** (B. F. FINKEL, B.Sc.)—Give Newton's mechanical method of describing the cissoid, with complete demonstration.

*Solution by* E. M. LANGLEY, M.A.

Newton, in his *Arithmetica Universalis*, says,—"The Ancients taught how to find two mean proportionals by the *cissoid;* but nobody that I know of hath given a good manual description of this curve. Let AG be the diameter and F the centre of a circle to which the *cissoid* belongs. At the point F let the perpendicular FD be erected and produced *in infinitum*, and let FG be produced to P so that FP may be equal to the diameter of the circle. Let the rectangular ruler PED be moved so that the leg EP may always pass through the point P and the other leg ED must be always equal to the diameter AG or FP with its end D always moving in the line FD; and the middle point C of this leg will describe the cissoid." (Wilder's edition.)




Because DP, PF = DP, DE; therefore right-angled triangles PFD, PED are congruent; therefore EF is parallel to PD, and PF, ED equally inclined to PD; therefore GC is parallel to PD and EF. Let it meet FD in H, the tangent at A in R, and the parallel to ED through F in Q.

Then FQ = EC = FG; therefore Q lies on the circle. Also, because CD is equal and parallel to FQ, therefore HC = HQ; but HR = HG, therefore CR = QG; therefore C lies on a cissoid with the circle AQG for generating circle. [See GREGORY's *Exs. in Diff. Calc.*, p. 130 (18461)].

**9490.** (Professor SCHOUTE.) — Two non-intersecting lines are the directors of a congruency (1, 1). Show that the locus of the axes of the complexes of the first order passing through the congruency is a ruled surface of the third order, the double line of which is the shortest distance of the two directors, while its simple line, that is no generator, is the line at infinity common to all the planes parallel to both the directors.

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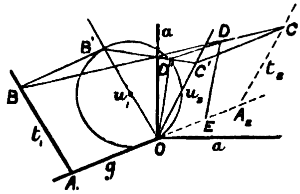
*Solution by the* PROPOSER.




As the axis of a linear complex is the line perpendicular to all the rays of the complex that meet it, and the directors of a congruency (1, 1) are conjugated to one another with reference to every linear complex passing through the congruency, the question can be stated in this form:

"Given two non-intersecting lines $A_1$ and $t_2$. To find the locus of the line $l$, that meets orthogonally the generators of the regulus $(t_1, t_2, l)$ determined by the three directors $t_1, t_2, l$."

The regulus $(t_1, t_2, l)$ is required to be a rectangular hyperbolical paraboloid, for the generators are parallel to the planes perpendicular to $l$, &c. Therefore $l$ meets orthogonally the line $g$ of shortest distance between $t_1$ and $t_2$.

Project the given lines $t_1$ and $t_2$ as $u_1$ and $u_2$ on the plane $\alpha$, perpendicular to the shortest distance $A_1A_2$ in its mid-point O. Then evidently the bisectors $a$ of the angles formed by $u_1$ and $u_2$ are the only lines in $\alpha$ satisfying the given conditions. And it may be proved as follows that through every point of $A_1A_2$ has two such lines.

We consider a plane parallel to $A_1A_2$, that meets $t_1$ in B, $t_2$ in C, and $\alpha$ in B′C′, the join of the projections B′ and C′ of B and C on $\alpha$. We seek the point D of BC, the projection of which on $\alpha$ is the foot D′ of the perpendicular through O on B′C′. Then the parallel DE to D′O is a generator of the ruled surface in question. For DE is the shortest distance between $A_1A_2$ and BC, and cuts orthogonally all the generators of the regulus $(t_1, t_2, \text{DE})$.

Now
$$\frac{A_1E}{A_1A_2} = \frac{BD}{BC} = \frac{B'D'}{B'C'};$$
so, when E is given, the last ratio is given too. And then D′ is one of the two points common to the circle on OB′ as diameter and the line parallel to $u_2$ that corresponds to this ratio.

Every plane $A_1A_2D$ through $A_1A_2$ contains one line DE that satisfies the conditions of the problem, as it admits one plane through BB′ orthogonal on it. Therefore every plane $A_1A_2D$ cuts the ruled surface in question in a curve of the third order consisting of the double line $A_1A_2$ and one generator; this proves the surface to possess the properties allotted to it in the question itself.

The locus of the axes of the linear complexes passing through a given congruency (1, 1) has been deduced analytically by Pluecker (*Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als*

*Raumelement*, Leipzig, Teubner, 1868, No. 86), who further illustrates the relation between the complexes and the congruency by means of the characteristic curve, obtained by measuring on every axis from $A_1A_2$ to either side a segment proportional to the parameter of the complex.

[For further study of the surface in question, see BALL's *Theory of Screws*, and *Bulletin of the Soc. Math. de France*, t. XIV., 1886.]

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**9986.** (Professor DÉPREZ.)—Soit $\beta$ l'angle compris entre la médiane et la symédiane issues du sommet B d'un triangle ABC, rectangle en A ; soit $\gamma$ l'angle compris entre la médiane et la symédiane partant de C. Démontrer la relation $\cot\beta\cot\gamma - 1 = 12\,(a/h)^2$, $h$ étant la hauteur menée par A.

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*Solution by* Professors BHATTACHARYA, BEYENS ; *and others*

The equations to the medians and symmedians from B and C are

$$ax - cz = 0,\quad cx - az = 0,\quad ax - by = 0,\quad bx - ay = 0 \;\ldots\ldots\; (1, 2, 3, 4)\;;$$

from (1), (2), we get $\qquad \cot\beta = c\,(3a^2+c^2)/b\,(c^2-a^2)$,

and from (3), (4), $\qquad \cot\gamma = b\,(3a^2+b^2)/c\,(b^2-a^2)$ ;

whence $\qquad \cot\beta\,.\cot\gamma - 1 = 12\,(b^2+c^2)^2/b^2c^2 = 12\,(a/h)^2$,

since $\qquad h = bc/a$, and $a^2 = b^2+c^2$.

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**10164.** (W. J. C. SHARP, M.A.)—Show that

$$\Delta^r(0\,!)\ \text{or}\ r\,!\left\{1-1+\frac{1}{1\,.\,2}-\frac{1}{1\,.\,2\,.\,3}+\ldots+(-1)^r\frac{1}{r\,!}\right\}$$
$$=(-1)^r\left\{1-r+r\,(r-1)-r\,(r-1)\,(r-2)+\ldots+(-1)^r\,r\,(r-1)\ldots3\right\}.$$

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*Solution by* R. KNOWLES, B.A. ; R. H. W. WHAPHAM ; *and others.*

Multiplying the left-hand term by $r\,!$ it becomes

$$3\,.\,4\,\ldots\,r - 4\,.\,5\,\ldots\,r + \ldots + (r-1)\,r - r + (-1)^r,$$

and writing this in the reverse order it becomes the right-hand term.

[This question originated from a statement, in TODHUNTER's *History of the Theory of Probability*, that LAMBERT had shown that $A_r \equiv \Delta^r(0\,!)$ satisfies the equation of differences $A_r = rA_{-1} + (-1)^r$, the solution of which question gives the second form for $\Delta^r(0\,!)$. Mr. SHARP is the more disposed to draw attention to this because he believes that this function will acquire increased importance and repay study.]

**9765.** (ARTEMAS MARTIN, LL.D.)—An urn contains 24 balls. The letter A is stamped on 8 of these balls at random, and the letter B is stamped at random on 6 of the balls. A ball is drawn from the urn at random. Find (1) the chance that the ball is not lettered, (2) the chance that it contains the letter A only, (3) the chance that it contains the letter B only, (4) the chance that it contains both letters.

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*Solution by* J. C. ST. CLAIR.

The chances are as given hereunder:—

$$(1)\quad \left(1-\frac{8}{24}\right)\left(1-\frac{6}{24}\right) = \frac{2}{3}\times\frac{3}{4} = \frac{1}{2},$$

$$(2)\quad \frac{8}{24}\times\left(1-\frac{6}{24}\right) = \frac{1}{3}\times\frac{3}{4} = \frac{1}{4},$$

$$(3)\quad \frac{6}{24}\times\left(1-\frac{8}{24}\right) = \frac{1}{4}\times\frac{2}{3} = \frac{1}{6},$$

$$(4)\quad \frac{8}{24}\times\frac{6}{24} = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}.$$

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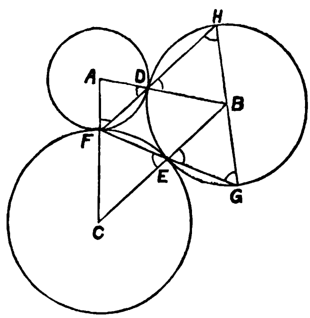
**9843.** (Professor MAYON.)—Soient A, B, C trois circonférences deux à deux tangentes aux points D, E, F; les droites FD, FE rencontrent B en des points H, G; démontrer que PQ passe par le centre de B, et qu'elle est parallèle à la ligne des centres des circonférences A et B.




*Solution by* A. B. EVANS; Prof. SARKAR; *and others.*

Join CEB, ADB and HB, BG

$\angle$ AFD = DDF = HDB = DHB,

therefore HB is parallel to AC,

$\angle$ CFE = FEC = BEG = EBG,

therefore BG is parallel to AC, therefore HG passes through B and is parallel to AC.

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**10110.** (Professor BHATTACHARYA.)—Show that the value of $\int dS/P$ for all the points of the surface of the ellipsoid represented by the equation

$$x^2/a^2+y^2/b^2+z^2/c^2 = 1 \text{ is } \tfrac{4}{3}\pi\,(b^2c^2+c^2a^2+a^2b^2)/abc.$$

*Solution by* D. EDWARDES, B.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

$$\iint \frac{dV}{dn}\, dS = \iiint \left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right) dx\, dy\, dz.$$

Let $V = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$; then

$$\frac{dV}{dn} = \frac{dV}{dx}\frac{dx}{dn} + \&c. = p\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) = \frac{1}{p};$$

therefore $$\iint \frac{dS}{p} = \iiint \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) dx\, dy\, dz$$

$$= \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \frac{\phi}{3} \pi abc = \text{result.}$$

---

**3304.** (Professor CAYLEY, F.R.S.)—The coordinates $x$, $y$, $z$ being proportional to the perpendicular distances from the sides of an equilateral triangle, trace the curve $(y-z)\, x^{\frac{1}{2}} + (z-x)\, y^{\frac{1}{2}} + (x-y)\, z^{\frac{1}{2}} = 0.$

*Solution by* Professor NASH, M.A.

The curve $(y-z)\sqrt{x} + \&c. = 0$ obviously consists of the three lines $y = z$, $z = x$, $x = y$. Clearing the equation of radicals, the sextic equation is easily seen to be the square of

$$x \cos A\, (y^2 - z^2) + y \cos B\, (z^2 - x^2) + z \cos C\, (x^2 - y^2) = 0,$$

a cubic which passes through the vertices A, B, C, the orthocentre L, the circumcentre O, the in- and ex-centres I, $I_a$, $I_b$, $I_c$, and the intersections $\alpha$, $\beta$, $\gamma$ of OA, OB, OC with the opposite sides.

The tangents at A, B, C, O intersect in L, those at I, $I_a$, $I_b$, $I_c$ in O, and those at $\alpha$, $\beta$, $\gamma$, L in another point also lying on the curve.

[Mr. J. GRIFFITHS remarks that the curve in question is a particular case of that which presents itself in the following theorem, communicated by him to Professor CAYLEY (with a demonstration) several years ago:—

The locus of a point $(x, y, z)$ such that its pedal circle (that is, the circle which passes through the feet of the perpendiculars drawn from the point in question upon the sides of the triangle of reference) touches the nine-point circle, is the sextic

$$\Sigma \left\{x \cos^2 A\, (y \cos B - z \cos C)\, (y \cos C - z \cos B)\right\}^{\frac{1}{2}} = 0,$$

which has been shown (Vol. VIII., p. 35) to consist of two coincident cubic curves given by the common equation $\Sigma x \cos A\, (y^2 - z^2) = 0$.]

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**10071.** (Professor DÉPREZ.)—Soient G, I, K le centre de gravité, le centre du cercle inscrit et le point de Lemoine du triangle ABC. Démontrer que

$$\text{GIK} : \text{ABC} = \left\{a^2(b-c) + b^2(c-a) + c^2(a-b)\right\} / 3\,(a+b+c)(a^2+b^2+c^2).$$

*Solution by* R. KNOWLES, B.A.; Professor MATZ; *and others.*

The coordinates of G, I, K are respectively (omitting $2\Delta$), $1/3a$, $1/3b$, $1/3c$; each $1/(a+b+c)$, $a/(a^2+b^2+c^2)$, $b/(a^2+b^2+c^2)$, $c/(a^2+b^2+c^2)$, and

$$\text{IK} = abc\left\{(a-b)^2+(a-c)^2+(b-c)^2+2(a-b)(a-c)\cos A\right.$$
$$\left.-2(a-b)(b-c)\cos B+2(a-c)(b-c)\cos C\right\}^{\frac{1}{2}} / (a+b+c)(a^2+b^2+c^2)$$
$$= abcd / (a+b+c)(a^2+b^2+c^2) \text{ suppose,}$$

and its equation is $(b-c)\,\alpha-(a-c)\,\beta+(a-b)\,\gamma = 0$;
we find length of perpendicular from G on IK

$$= 2\Delta\left\{a^2(b-c)+b^2(c-a)+c^2(a-b)\right\} / 3abcd,$$

$\therefore$ $\text{GIK} = 2\Delta\left\{a^2(b-c)+b^2(c-a)+c^2(a-b)\right\} / 3(a+b+c)(a^2+b^2+c^2)$;
hence the result as in the question.

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**10147.** (Professor HUDSON.)—If A, B, B be the angles of an isosceles triangle, $2\sin^2(A-B)(2-\cos A) = (\sin^2 A+2\sin^2 B)(1-8\cos A\cos^2 B)$.

---

*Solution by* W. J. GREENSTREET, M.A.; R. KNOWLES, B.A.; *and others.*

$$2\sin^2(A-B) = 2(a^2-b^2)^2\sin^2 B / b^4 \quad\text{..................(1)},$$
$$2-\cos A = (a^2+2b^2)/2b^2 \quad\text{..........................(2)},$$
$$\sin^2 A+2\sin^2 B = (a^2+2b^2)\sin^2 B / b^2 \quad\text{..................(3)},$$
$$1-8\cos A\cos B = 9-(a^2+2b^2)/R^2 = (a^2-b^2)^2/b^4 \quad\text{.........(4)},$$

since $\quad a = 2b\cos B$ and $R^2 = b^4/(4b^2-a^2)$,

therefore $\quad (1)\times(2) = (3)\times(4)$.

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**10282.** (D. BIDDLE.)—Six balls of different colours, but otherwise indistinguishable, are placed in a bag. One is drawn, and its colour having been recorded, is replaced. The process is subsequently repeated five times, the drawer receiving a sovereign if he draw a new colour, but forfeiting a sovereign if he draw one already drawn. Prove that, if the drawer pay a sovereign to begin with each round, the bank secures on the average something less than sixpence, or about $4\frac{1}{2}d$.

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*Solution by* A. J. PRESSLAND, B.A.; G. G. STORR, M.A.; *and others.*

Suppose the balls denoted by $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $i$, of which $\alpha$ is first drawn; then we have the annexed scheme [given on p. 82], from which it appears that the Bank gains

$$1-\frac{7630}{7776} \text{ of } £1 = \frac{146}{7776}\times 240 \text{ pence} = 4\tfrac{4}{8}\tfrac{1}{1}d.$$

| Type. | Number = | | Value of Type. | + | − |
|---|---|---|---|---|---|
| $\alpha^5$ | $1$ | 1 | −5 | | 5 |
| $\alpha^4\beta$ | $5 \cdot \frac{5!}{4!}$ | 25 | −3 | | 75 |
| $\alpha^3\beta^2$ | $5 \cdot \frac{5!}{3!\,2!}$ | 50 | −3 | | 150 |
| $\alpha^3\beta\gamma$ | $\frac{5.4}{1.2} \cdot \frac{5!}{3!}$ | 200 | −1 | | 200 |
| $\alpha^2\beta^3$ | $5 \cdot \frac{5!}{2!\,3!}$ | 50 | −3 | | 150 |
| $\alpha^2\beta^2\gamma$ | $5.4 \cdot \frac{5!}{2!\,2!}$ | 600 | −1 | | 600 |
| $\alpha^2\beta\gamma\delta$ | $\frac{5.4.3}{1.2.3} \cdot \frac{5!}{2!}$ | 600 | +1 | 600 | |
| $\alpha\beta^4$ | $5.5$ | 25 | −3 | | 75 |
| $\alpha\beta^3\gamma$ | $5.4 \cdot \frac{5!}{3!}$ | 400 | −1 | | 400 |
| $\alpha\beta^2\gamma^2$ | $\frac{5.4}{1.2} \cdot \frac{5!}{2!\,2!}$ | 300 | −1 | | 300 |
| $\alpha\beta^2\gamma\delta$ | $5 \cdot \frac{4.3}{1.2} \cdot \frac{5!}{2!}$ | 1800 | +1 | 1800 | |
| $\alpha\beta\gamma\delta\epsilon$ | $5.5!$ | 600 | +3 | 1800 | |
| $\beta^5$ | $5$ | 5 | −3 | | 15 |
| $\beta^4\gamma$ | $5.4 \cdot \frac{5!}{4!}$ | 100 | −1 | | 100 |
| $\beta^3\gamma^2$ | $5.4 \cdot \frac{5!}{3!\,2!}$ | 200 | −1 | | 200 |
| $\beta^3\gamma\delta$ | $5 \cdot \frac{4.3}{1.2} \cdot \frac{5!}{3!}$ | 600 | +1 | 600 | |
| $\beta^2\gamma^2\delta$ | $5 \cdot \frac{4.3}{1.2} \cdot \frac{5!}{2!\,2!}$ | 900 | +1 | 900 | |
| $\beta^2\gamma\delta\epsilon$ | $5.4 \cdot \frac{5!}{2!}$ | 1200 | +3 | 3600 | |
| $\beta\gamma\delta\epsilon\iota$ | $1.5!$ | 120 | +5 | 600 | |
| | | | | +9900 | −2270 |
| | | | | 2270 | |
| | Total ... | 7776 | | 7630 | |

**10134.** (E. M. LANGLEY, M.A.)—Deduce the existence of the Brocard-points from that of the symmedian point.

*Solution by* J. J. BARNIVILLE; E. RUTTER; *and others.*

Through P, the symmedian point, draw parallels to the sides, meeting them in DD′EE′FF′; let O be the point in DEF corresponding to P in D′E′F′, and let O′ be the point in D′E′F′ corresponding to P in DEF, then O, O′ are the Brocard-points.

**10138.** (JOHN J. BARNIVILLE.)—Sum to $n$ terms the series

$$3+8+16+28+46+\&c.$$

*Solution by* D. BIDDLE; H. J. WOODALL; *and others.*

The first term $=1+2$, and the successive differences are $1+4$, $2+6$, $4+8$, $8+10$, &c. Therefore the $n^{\text{th}}$ term $=2^{n-1}+2\Sigma_1^n x = 2^{n-1}+n\,(n+1)$, and the sum of the series $=2^n-1+\frac{1}{3}n\,(n+1)(n+2)$.

**10046.** (MAURICE D'OCAGNE.)—Soit ABC un triangle inscrit dans une hyperbole équilatère ayant pour centre le milieu O de BC; démontrer que (1) la tangente en A à cette hyperbole est symédiane du triangle ABC; (2) si le diamètre OM perpendiculaire à BC coupe AB en M, et que H soit le pied de la perpendiculaire abaissée de M sur AC, les axes de l'hyperbole sont les bissectrices de l'angle COH; (3) si un cercle et une hyperbole équilatère sont concentriques, les tangentes à l'hyperbole aux extrémités d'un de leurs diamètres communs sont perpendiculaires à l'autre diamètre commun.

*Solution by* R. KNOWLES, B.A.; Prof. NILKANTHA SARKAR; *and others.*

(1) Let the tangent at A meet BC in P, and the coordinates of ABC and the equations to BC and the tangent A be respectively

$m\,a^2/m,\ m_1\,a^2/m_1,\ -m_1\,-a^2/m_1,\ a^2x-m^2y=0,\ a^2x+m^2y=2a^2m\ldots(1,2)$,

whence AB : AC $=(m-m_1):(m+m_1)$; from (1), (2), the coordinates of P are $2mm_1^2/(m^2+m_1^2),\ 2a^2m/(m^2+m_1^2)$,

therefore $\quad$ PB : PC $=(m-m_1)^2:(m+m_1)^2=$ AB$^2$ : AC$^2$,

therefore AP is the symmedian from A.

(2) The equations to BA and OM are

$$a^2x+mm_1y=a^2\,(m+m_1),\quad m_1^2x+a^2y=0,$$

whence the coordinates of M are

$$a^4\,(m+m_1)/(a^4-mm_1^3),\quad -a^2m^2\,(m+m_1)/(a^4-mm_1^3);$$

and the equations to MH and AC are

$$mm_1(a^4 - mm_1^3)x + a^2(a^4 - mm_1^3)y + a^4m(m^2 - m_1^2) = 0,$$

$$a^2x - mm_1y = a^2(m - m_1),$$

therefore the coordinates of H are

$$a^4(m - m_1)/(a^4 - mm_1^3), \quad a^2m^2(m - m_1)/(a^4 - mm_1^3),$$

and the equation to OH $\qquad m^2x - a^2y = 0$ ..................................(3),
it is easily seen that the lines (2), (3) make equal angles with $x \pm y = 0$, that is, the axes bisect the angle COH.

(3) The tangent $a^2x + m^2y = 2a^2m$ is at right angles to $m^2x - a^2y = 0$, the equation to the other common diameter.

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**8246.** (By Professor Wolstenholme, Sc.D.)—In a cubic with three real asymptotes and an acnode, prove that the area between any two asymptotes and the corresponding infinite branch is one-third of the area of the triangle formed by the asymptotes.

*Solution by the* Proposer.

Since by an orthogonal projection, which does not alter the ratios of the areas, we can make the three asymptotes form an equilateral triangle, in which case the three areas are obviously equal, they will be equal in all cases. Taking two of the asymptotes as coordinate axes, we may take the equation to be $\qquad xy\left(3 + \frac{x}{a} + \frac{y}{b}\right) = ab,$

the acnode being the point $x = -a$, $y = -b$. The area between the coordinate axes and the corresponding infinite branch will then be $\sin\omega \int_0^\infty y\,dx$, where $y$ has the positive value given by the equation, and $\omega$ is the angle between the coordinate axes. Hence

$$2\frac{y}{b} = -3 - \frac{x}{a} + \sqrt{\left(3 + \frac{x}{a}\right)^2 + 4\frac{a}{x}} = -3 - \frac{x}{a} + \frac{(x+a)\sqrt{x+4a}}{a\sqrt{x}},$$

and the area $= \dfrac{b}{2a}\sin\omega \displaystyle\int_0^\infty \left((x+a)\sqrt{\frac{x+4a}{x}} - 3a - x\right)dx$

$$= \frac{b}{2a}\sin\omega\left[\frac{\sqrt{x}(x+4a)^{\frac{3}{2}}}{2} - 3ax - \frac{x^2}{2}\right]_0^\infty$$

$$= \frac{b}{4a}\sin\omega\left[x^2\left(1 + \frac{4a}{x}\right)^{\frac{3}{2}} - 6ax - x^2\right]_0^\infty$$

$$= \frac{b}{4a}\sin\omega\left[x^2\left(1 + \frac{6a}{x} + \frac{6a^2}{x^2} + \dots\right) - 6ax - x^2\right]_{x=\infty}$$

$$= \frac{3ab}{2}\sin\omega = \tfrac{1}{3}\text{ area formed by the three asymptotes.}$$

There is no difficulty in calculating the other two areas directly, but it

would be clearly a work of supererogation. The equation of the curve referred to areal coordinates measured on the triangle formed by the asymptotes is $(x+y+z)^3 = 27xyz$; and of all the cubics represented by $(x+y+z)^3 = kxyz$, this is the only one whose area can be found without the use of Elliptic Functions. I never thought of trying whether any such case existed, and only discovered it by happening to notice that the curve $x^{2n+1}+y^{2n+1} = a\,x^ny^n$ gives this cubic when $n = -\frac{1}{3}$, and the sectorial area of this curve for any value of $n$ is easily found by ordinary integration, using polar coordinates. So, if $n = \frac{1}{3}$, we get the quintic $x^5 + y^5 + 3ax^2y^2 = a^3xy$, and the area of the loop of this curve = area between the curve and its one real asymptote = $\frac{3}{10}a^2$. No doubt many other areas may be similarly determined.

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**2632.** (N'IMPORTE.) — Prove that (1) $1.2.3\ldots n < 2^{\frac{1}{2}n(n-1)}$; and (2) $\dfrac{27a^2b^2}{(a+b)^3} < 4a$ or $4b$, when $a$ and $b$ are both positive.

*Solution by* G. E. CRAWFORD, B.A.

(1) $$2^n = (1+1)^n = 1+n+\ldots > 1+n,$$
if $n$ be 2 or $>2$. Now $2^0 = 1,\ 2^1 = 2,\ 2^2>3,\ 2^3>4,\ \ldots\ldots\ 2^{n-1}>n$.

Therefore $$1.2.3\ldots n < 2^{1+2+3+\ldots n-1} < 2^{\frac{1}{2}n(n-1)},$$
except when $n = 1$ or $2$.

(2) Taking three quantities $a,\ \frac{1}{2}b,\ \frac{1}{2}b$, the A. M. and G. M. are $\frac{1}{3}(a+b)$, $(\frac{1}{4}ab^2)^{\frac{1}{3}}$; therefore $\frac{1}{27}(a+b)^3 > \frac{1}{4}ab^2$, therefore $\dfrac{27a^2b^2}{(a+b)^3} < 4a$ and also $< 4b$ by symmetry.

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**2214.** (Professor WOLSTENHOLME, M.A., Sc.D.)—The polar plane of a fixed point (X, Y, Z) being taken with respect to one of a series of confocal quadrics $$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$
prove that the straight line along which this polar plane touches its envelope is normal to another confocal of the system
$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1,$$
where $$\frac{(a^2+\mu)(a^2+\lambda)^4}{X^2(a^2-b^2)^2(a^2-c^2)^2} + \frac{(b^2+\mu^2)(b^2+\lambda)^4}{Y^2(b^2-c^2)^2(b^2-a^2)^2} + \frac{(c^2+\mu)(c^2+\lambda)^4}{Z^2(c^2-a^2)^2(c^2-b^2)^2} = 1$$
at the point (X′, Y′, Z′) such that
$$\frac{XX'}{(a^2+\lambda)^2(a^2+\mu)(b^2-c^2)} = \frac{YY'}{(b^2+\lambda)^2(b^2+\mu)(c^2-a^2)}$$
$$= \frac{ZZ'}{(c^2+\lambda)^2(c^2+\mu)(a^2-b^2)} = \frac{1}{(b^2-c^2)(a^2-b^2)(a^2-c^2)}.$$

*Solution by* W. J. Curran Sharp, M.A.

The polar plane of (X, Y, Z) with respect to

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$$

is
$$\frac{Xx}{a^2+\lambda}+\frac{Yy}{b^2+\lambda}+\frac{Zz}{c^2+\lambda}=1,$$

which touches its envelope along its intersection with

$$\frac{Xx}{(a^2+\lambda)^2}+\frac{Yy}{(b^2+\lambda)^2}+\frac{Zz}{(c^2+\lambda)^2}=0,$$

*i.e.*, along the line $\frac{A}{x'}(x-x')=\frac{B}{y'}(y-y')=\frac{C}{z'}(z-z')$ ...................(1),

if
$$\frac{Xx'}{(a^2+\lambda)^2}\cdot\frac{(a^2-b^2)(a^2-c^2)}{A}=\frac{Yy'}{(b^2+\lambda)^2}\cdot\frac{(b^2-c^2)(b^2-a^2)}{B}$$
$$=\frac{Zz'}{(c^2+\lambda)^2}\cdot\frac{(c^2-a^2)(c^2-b^2)}{C}=\frac{a^2-b^2}{A-B}=\frac{b^2-c^2}{B-C}=1 \text{ .........(2).}$$

Therefore A : B : C :: $a^2+\mu : b^2+\mu : c^2+\mu$, and

$$\frac{Xx'}{(a^2+\lambda)^2(a^2+\mu)(b^2-c^2)}=\frac{Yy'}{(b^2+\lambda)^2(b^2+\mu)(c^2-a^2)}=\frac{Zz'}{(c^2+\lambda)^2(c^2+\mu)(a^2-b^2)}$$
$$\equiv k \text{ say ...............(3);}$$

from equations (2), $\frac{Xx'}{(a^2+\lambda)^2(b^2-c^2)}-\frac{Yy'}{(b^2+\lambda)^2(c^2-a^2)}=\frac{1}{(b^2-c^2)(c^2-a^2)}$;

by (3), therefore, $k(a^2+\mu)-k(b^2+\mu)=\frac{1}{(b^2-c^2)(c^2-a^2)}$,

and
$$k=\frac{1}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}.$$

Now, if $\frac{x'^2}{a^2+\mu}+\frac{y'^2}{b^2+\mu}+\frac{z'^2}{c^2+\mu}=1$, the line (1) is the normal to

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}+\frac{z^2}{c^2+\mu}=1 \text{ at } (x', y', z'),$$

and, from (3),
$$k^2\left\{\frac{(b^2-c^2)^2(a^2+\lambda)^4(a^2+\mu)}{X^2}+\frac{(c^2-a^2)^2(b^2+\lambda)^4(b^2+\mu)}{Y^2}\right.$$
$$\left.+\frac{(a^2-b^2)^2(c^2+\lambda)^4(c^2+\mu)}{Z^2}\right\}=1,$$

and
$$\frac{(a^2+\mu)(a^2+\lambda)^4}{X^2(a^2-b^2)^2(a^2-c^2)^2}+\frac{(b^2+\mu)(b^2+\lambda)^4}{Y^2(b^2-c^2)^2(b^2-a^2)^2}+\frac{(c^2+\mu)(c^2+\lambda)^4}{Z^2(c^2-a^2)^2(c^2-b^2)^2}=1.$$

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**10084.** (H. L. Orchard, M.A.)—Solve

$$x^8+(x^2-x)^4+(x^2-2x)^4+(x^2-3x+2)^4+9(x-1)^4+7(x-2)^4$$
$$+16x^4+63=0.$$

*Solution by* H. J. WOODALL; Professor NILKANTHA SARKAR; *and others.*

Equation $\equiv \{x^4+(x-1)^4+7\}\ \{x^4+(x-2)^4+9\} = 0$;
thus there is evidently no real solution.

To solve $x^4+(x-1)^4+7=0$, put $x-\frac{1}{2}=y$; then

$$y^2 = -\tfrac{3}{4} \pm (-3)^{\frac{1}{2}}, \text{ and } x = \tfrac{1}{2} \pm \{-3 \pm (-3)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

To solve $x^4+(x-2)^4+9=0$, put $x-1=y$; then

$$y^2 = -3 \pm (\tfrac{7}{2})^{\frac{1}{2}}, \text{ and } x = y+1 = 1 \pm \{-3 \pm (\tfrac{7}{2})^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

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**2814.** (The late MATTHEW COLLINS, B.A.)—Prove that the common difference of three rational square integers in arithmetical progression can never be equal to 17.

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*Note by* ARTEMAS MARTIN, LL.D.

It is well known that $(2rs-r^2+s^2)^2$, $(r^2+s^2)^2$, $(2rs+r^2-s^2)^2$, are *general* expressions for three *square* numbers in arithmetical progression, where $r$ and $s$ may be any unequal whole numbers, $r > s$. Let $r=2$, $s=1$; then 1, 25, and 49 are three square numbers in arithmetical progression, and the least known. See *Gill's Angular Analysis*, p. 14; *Maynard's Key to Bonnycastle's Introduction to Algebra*, p. 113; *Tyson's Key to the same*, p. 186. The common difference is

$$(r^2+s^2)^2-(2rs-r^2+s^2)^2 = (2rs+r^2-s^2)^2-(r^2+s^2)^2 = 4rs\,(r^2-s^2).$$

It is therefore evident that the common difference is *always* divisible by 4, and consequently *can never* be $= 17$, nor any other prime number.

[On p. 160 of Vol. XLIX., Mr. CHRISTIE has obtained other expressions for three square numbers in arithmetical progression, and on pp. 161, 175 of the same volume he gives further developments bearing on the subject.]

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**10020.** (SARAH MARKS, B.Sc.)—If D, E, F are the points of contact of the inscribed circle with the sides BC, CA, AB respectively, show that, if the squares of AD, BE, CF are in arithmetical progression, then the sides of the triangle are in harmonical progression.

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*Solution by* W. J. GREENSTREET, M.A.; H. W. SEGAR; *and others.*

$$\text{BD} = s-b; \quad \text{therefore } \text{AD}^2 = c^2+(s-b)^2-2c\,(s-b)\cos \text{B},$$

$$\text{CD} = s-c; \quad \text{therefore } \text{AD}^2 = c^2+(s-c)^2-2b\,(s-c)\cos \text{C};$$

therefore, simplifying, $2\text{AD}^2 = a^2+b^2+c^2-2s^2+2bc\,(\cos \text{A}+\cos \text{B}+\cos \text{C})$.

Similarly $2\text{BE}^2 = a^2+b^2+c^2-2s^2+2ca\,(\cos \text{A}+\cos \text{B}+\cos \text{C})$,

$$2\text{CF}^2 = a^2+b^2+c^2-2s^2+2ab\,(\cos \text{A}+\cos \text{B}+\cos \text{C}).$$

If $\text{AD}^2+\text{CF}^2 = 2\text{BE}^2$, we get $ab+bc=2ca$, or $b = \dfrac{2ac}{c+a}$, *i.e.*, sides in H.P.

**10117.** (Professor CATALAN.)—Dans tout quadrilatère convexe, dont les angles sont A, B, C, D, démontrer que

$$\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\cos\tfrac{1}{2}D=\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+A);$$
$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C\sin\tfrac{1}{2}D=\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+A);$$
$$\sin\tfrac{1}{2}(A+B)\left[\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C\sin\tfrac{1}{2}D\right]$$
$$=\sin\tfrac{1}{2}(B+C)\left[\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C+\sin\tfrac{1}{2}D\sin\tfrac{1}{2}A\right];$$
$$\sin\tfrac{1}{2}(A+B)\left[\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\cos\tfrac{1}{2}D\right]$$
$$=\sin\tfrac{1}{2}(B+C)\left[\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C+\cos\tfrac{1}{2}D\cos\tfrac{1}{2}A\right];$$
$$2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+D)=-\sin\tfrac{1}{2}A\sin\left\{C+\tfrac{1}{2}(B+D)\right\}$$
$$+\sin\tfrac{1}{2}B\sin\tfrac{1}{2}(B+C)+\sin\tfrac{1}{2}D\sin\tfrac{1}{2}(C+D).$$

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*Solution by* G. E. CRAWFORD, B.A.; Rev. J. L. KITCHIN; *and others.*

Since $A+B+C+D=2\pi$, therefore

$$\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\cos\tfrac{1}{2}D=\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C\sin\tfrac{1}{2}D.$$
$$=\sin\tfrac{1}{2}(C+A)\sin\tfrac{1}{2}(B+C).$$

Thus $\quad\sin\tfrac{1}{2}(A+B)\left\{\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C\sin\tfrac{1}{2}D\right\}$

$$=\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+A)$$
$$=\sin\tfrac{1}{2}(B+C)\left\{\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C+\sin\tfrac{1}{2}D\sin\tfrac{1}{2}A\right\} \text{ by symmetry.}$$

and the fourth relation in precisely the same way.

Lastly

$$-\sin\tfrac{1}{2}A\sin\left\{C+\tfrac{1}{2}(B+D)\right\}+\sin\tfrac{1}{2}B\sin\tfrac{1}{2}(B+C)+\sin\tfrac{1}{2}D\sin\tfrac{1}{2}(C+D)$$
$$=-\sin\tfrac{1}{2}A\sin\left\{C+\tfrac{1}{2}(B+D)\right\}+\sin\tfrac{1}{2}B\sin\tfrac{1}{2}(A+D)+\sin\tfrac{1}{2}D\sin\tfrac{1}{2}(A+B)$$
$$=2\cos\tfrac{1}{2}A\left\{\sin\tfrac{1}{2}A\sin\tfrac{1}{2}C+\sin\tfrac{1}{2}B\sin\tfrac{1}{2}D\right\}$$
$$=2\cos\tfrac{1}{2}A\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+D) \text{ by (1).}$$

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**9500.** (Professor KALIPADA BASÛ, M.A.)—Two systems of three forces (P, Q, R), and (P′, Q′, R′) act along the sides of a triangle ABC. Find the condition (1) that the resultants may be parallel, (2) that they may be perpendicular.

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*Solution by* Prof. ABINASH BASÛ, M.A.; J. MACMAHON, M.A.; *and others.*

If $(x, y, z)$ be the coordinates of a point on the resultant of (P, Q, R), then by taking moments $\quad Px+Qy+Rz=0$ .............................. (1),
this is the equation in trilinears to the resultant of (P, Q, R).
Similarly, $\quad P'x+Q'y+R'z=0$ .............................. (2)
is the other equation. Since these are to be parallel, the line at infinity passes through them; therefore

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ \sin A, & \sin B, & \sin C \end{vmatrix}=0$$ is the condition of parallelism; also the condition of perpendicularity of (1) and (2) is

$$PP'+QQ'+RR'-(PQ'+P'Q)\cos A-\ldots=0,$$

as usually given.

**10205.** (F. A. Tarleton, LL.D.)—A billiard player plays full, with a strong top twist, against a ball touching a cushion, which makes a small angle with the line of direction of his blow; find an expression for the greatest distance of the striker's ball from the cushion, after hitting it, and show that approximately the ball runs along the cushion.

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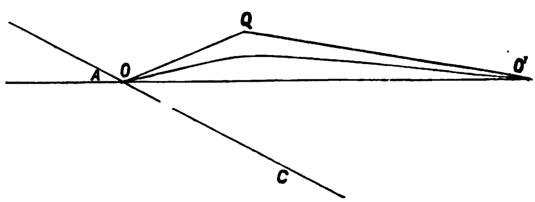
*Solution by the* Proposer.

Let A be the angle that the cushion makes with the line of direction of the motion of the striker's ball, V the velocity of its centre when it impinges against the other ball, and $v$ the velocity of the same point after the collision; then it is easy to see that $v = \mathrm{V}\dfrac{\cos^2 \mathrm{A} - e'}{1 + \cos^2 \mathrm{A}}$, where $e'$ is the coefficient of restitution for the two balls. After this collision the striker's ball runs on and hits the cushion with a velocity $v$ and rebounds, the velocities of its centre $u_1$ and $v_1$ parallel and perpendicular to the cushion being given by the equations $u_1 = v\cos \mathrm{A}$, $v_1 = ev \sin \mathrm{A}$, where $e$ is the coefficient of restitution for the cushion.

When impinging against the cushion the ball has an angular velocity $\Omega$ round an axis perpendicular to the line of motion of its centre. If impulsive friction be neglected, this angular velocity may be regarded as unaltered by the impact, so that after rebounding the angular velocities $\varpi_1$ and $\sigma_1$ of the ball round axes parallel and perpendicular to the cushion are given by the equations $\varpi_1 = \Omega \sin \mathrm{A}$, $\sigma_1 = \Omega \cos \mathrm{A}$. The ball now (*Dynamics*, Art. 278, Ex. 1) describes a parabola under the influence of a constant acceleration $\mu g$, where $\mu$ is the coefficient of friction, whose direction makes an angle $a_1$ with the cushion, $a_1$ being given by the equation

$$\tan a_1 = \frac{v_1 + a\varpi_1}{a\sigma_1 - u_1} = \frac{ev + a\Omega}{a\Omega - v} \tan \mathrm{A}.$$




Now $\sigma_1$ is large, being nearly equal to $\Omega$, and $u_1$ is small, being less than $v$, which is small. Hence $u_1 - a\sigma_1$ is negative, and the acceleration $\mu g$ tends therefore (*Dynamics*, Art. 278, Ex. 1) to increase the velocity of the centre parallel to the cushion, but to diminish its velocity perpendicular to the cushion, and the path of the centre is therefore represented by the parabola in the accompanying figure, in which OO′ represents a parallel to the line of the cushion, OC the direction of the constant acceleration $\mu g$, and OQ the direction of the initial motion of the centre of the ball after rebounding from the cushion.

If $V_1$ be the initial velocity of the centre after the rebound of the ball, we have $$V_1^2 = v^2(\cos^2 A + e^2 \sin^2 A),$$

and by the formula for parabolic motion (*Dynamics*, Art. 50), if $t_1$ be the time of going from O to O′, and $A_1$ the angle QOO′, we have also $t_1 = \dfrac{2V_1 \sin A_1}{\mu g \sin \alpha_1}$; whence it is easy to see that the greatest distance of the ball from the cushion in going from O to O′ is $\dfrac{V_1^2}{2\mu g} \dfrac{\sin^2 A_1}{\sin \alpha_1}$.

If $B_1$ be the angle QO′O at which the ball strikes the cushion at O′, and $A_2$ the angle at which it rebounds, we have (Art. 56),

$$\cot B_1 = \cot A_1 + 2 \cot \alpha_1,$$

also $\tan A_2 = e \tan B_1$. Again, if $\varpi_2$ and $\sigma_2$ be the angular velocities of the ball after the second rebound, and $u_2$ and $v_2$ the velocities of its centre, we have

$$a\varpi_2 = a\varpi_1 - \tfrac{5}{2}\mu g \sin \alpha_1 t_1, \quad a\sigma_2 = a\sigma_1 - \tfrac{5}{2}\mu g \cos \alpha_1 t_1,$$
$$u_2 = u_1 + \mu g \cos \alpha_1 t_1, \quad v_2' = v_1 - \mu g \sin \alpha_1 t_1, \quad v_2 = ev_2',$$
$$V_2^2 = u_2^2 + v_2^2, \quad \tan \alpha_2 = \frac{v_2 + a\varpi_2}{a\sigma_2 - u_2} = \tan \alpha_1 - \frac{(1-e)\, v_2'}{a\sigma_2 - u_2},$$

since $\dfrac{v_2' + a\varpi_2}{u_2 - a\sigma_2} = \dfrac{v_1 + a\varpi_1}{u_1 - a\sigma_1}$ (*Dynamics*, Art. 278, Ex. 1).

Hence we may conclude that each of the successive parabolic arcs described by the ball is flatter than the preceding one.

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**10290.** (Maurice d'Ocagne.)—Soient ABC un triangle rectangle en A, AH la hauteur issue de A, HK la perpendiculaire abaissée de H sur AB. CK coupe AH en I. Démontrer que la perpendiculaire abaissée de I sur AC coupe ce côté au même point que la symédiane issue de B.

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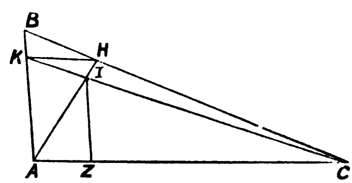
*Solution by* Rev. T. R. Terry, M.A.; R. Knowles, B.A.; *and others.*




Let IZ be the perpendicular on AC, and E the middle point of AC. If ZBA = $\alpha$ and EBA = $\beta$, we have to show that $\alpha + \beta = B$.

Now $AH = a \sin B \cos B$,

$AK = a \sin^2 B \cos B$,

$\therefore \quad IZ/ZC = \sin B \cos B$.

But $\quad IZ/AZ = \tan B$,

therefore $\qquad AZ = a\cos^2 B \sin B \,/\, (1+\cos^2 B)$,

therefore $\qquad \tan\alpha = \sin B \cos B \,/\, (1+\cos^2 B)$.

And $\quad \tan\beta = \frac{1}{2}\sin B \sec B$, therefore $\tan(\alpha+\beta) = \tan B$.

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**10220.** (Professor WOLSTENHOLME, M.A., Sc.D.)—A circle is described with its centre on the directrix of a given paral ola and diameter equal to the focal distance of the centre; prove that three of the common tangents to this circle and the parabola will form an equilateral triangle; and the centres of the escribed circles of any such equilateral triangle lie on the cubic $\qquad y^2(3x+7a)+4a^2(3a-x) = 0.$

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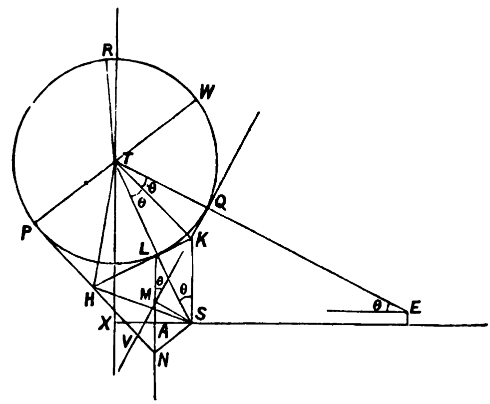
*Solution by* G. E. CRAWFORD, B.A.; Rev. T. GALLIERS, M.A.; *and others.*

The tangent at L to the circumcircle is clearly one common tangent. Let PN and VQ be two others and join as in Fig. Then

$$\angle\, WTQ = PVQ = HSK = HTK = \tfrac{1}{2}PTQ;$$




therefore $WTQ = \frac{1}{3}\pi$, and therefore $PVQ = \frac{1}{3}\pi$. Similarly, VQ and the other direct common tangent are inclined at angle $\frac{1}{3}\pi$.

Let E be the escribed centre, coordinates $x$, $y$, referred to X; therefore

$$x = TE\cos\theta = 4TQ\cos\theta = 4a\sec TSX\cos\theta = 4a\cos\theta\sec 3\theta \;\ldots(1),$$

$$y = TX - 4TQ\sin\theta = 2a\tan 3\theta - 4a\sin\theta\sec 3\theta \;\ldots\ldots\ldots\ldots(2).$$

From (1), $\quad \cos^2\theta = \dfrac{4a+3x}{4x}$, therefore $\sin^2\theta = \dfrac{x-4a}{4x}$,

and from (2), $\quad y\cos 3\theta - 2a\sin 3\theta + 4a\sin\theta = 0$,

therefore $\quad y\cos\theta\left(\dfrac{4a}{x}\right) - 2a\sin\theta\left(\dfrac{2x+4a}{x}\right) + 4a\sin\theta = 0,$

therefore $y \cos \theta = 2a \sin \theta$, whence $y^2(4a+3x) = 4a^2(x-4a)$,
or, referring to A as origin,

$$y^2(3x+7a) + 4a^2(3a-x) = 0.$$

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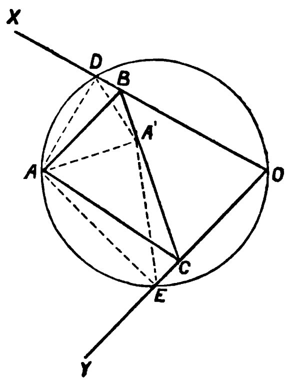
**10188.** (Professor NEUBERG.)—Un angle de grandeur constante tourne autour de son sommet A ; ses côtés rencontrent deux axes fixes OX, OY en B, C. Trouver (1°) l'enveloppe de BC, et celles des hauteurs BB′, CC′ ; (2°) les lieux décrits par les pieds A′, B′, C′ des hauteurs et par l'orthocentre H.

*Solution by* E. M. LANGLEY, M.A.




Since triangles ABB′, ACC′ are of constant species, B′, C′ describe straight lines, and BB′, CC′ envelope parabolas.

Draw AD, AE perpendicular to OX OY. Then, because A, D, B, A′ are concyclic, $\angle$ BA′D = BAD.
Similarly $\angle$ CA′E = CAE,
∴ $\angle$ BA′D + CA′E + BAC + XOY
= DAE + XOY = 2 rt. $\angle$'s,
∴ $\angle$ DA′E = BAC + XOY ;
hence A′ describes a circle, and BC envelopes a central conic with A for one focus. [See Vuibert's *Questions de Mathématiques*, pp. 89–91.]

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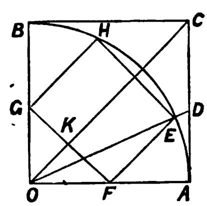
**10208.** (J. J. BARNIVILLE.)—Inscribe in a quadrant a square having two angles on the circumference.

*Solution by* Rev. T. GALLIERS, M.A. ; D. T. GRIFFITHS ; *and others.*




Let AOB be the quadrant; draw the square OACB ; bisect AC in D ; join OD, cutting the circumference in E ; join OC and draw EF parallel to OC, meeting OA in F ; then the square drawn on EF towards OC will be the square required. For, if $\angle$ FOE = $\theta$ and OE = $a$, $\sin\theta = 1/5^{\frac{1}{2}}$, therefore $\cos\theta = 2/5^{\frac{1}{2}}$,

$$\sin \text{FEO} = 1/(10)^{\frac{1}{2}} \text{ and } \text{FE} = 2^{\frac{1}{2}}a/5^{\frac{1}{2}},$$

therefore $\text{OF} = a/5^{\frac{1}{2}}$.

If FK be perpendicular to OC,

$$\text{FK} = \text{OF}/2^{\frac{1}{2}} = a/(10)^{\frac{1}{2}} = \tfrac{1}{2}\text{FE}.$$

**10245.** (H. W. Segar.)—Prove that the distance between the orthocentre of a triangle and the centre of the circumscribed circle is, in terms of the sides of the triangle,

$$\left\{a^2(a^2-b^2)(a^2-c^2)+b^2(b^2-c^2)(b^2-a^2)+c^2(c^2-a^2)(c^2-b^2)\right\}/(4\Delta)^2.$$

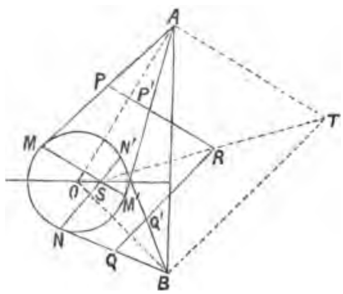
*Solution by* R. H. Whapham, B.A.; G. G. Storr, M.A.; *and others.*

If O be the circumcentre and P the orthocentre, we have

$$\begin{aligned} OP^2 &= R^2(1-8\cos A\cos B\cos C) \\ &= R^2\left\{a^6+b^6+c^6-a^4(b^2+c^2)-b^4(c^2+a^2)-c^4(a^2+b^2)+3a^2b^2c^2\right\}/a^2b^2c^2 \\ &= \left\{a^2(a^2-b^2)(a^2-c^2)+\ldots\right\}/(4\Delta)^2. \end{aligned}$$

**10233.** (Professor Moreau.)—On donne trois points fixes A, B, O. Le point O est le centre d'un cercle de rayon variable. Par les points A, B, on mène des tangentes AM, AM′, BN, BN′ au cercle variable; on joint les milieux P, P′ de AM, AM′, et les milieux Q, Q′ de BN, BN′. Lieu du point d'intersection des deux droites PP′ et QQ′.

*Solution by* Prof. P. H. Schoute; R. Knowles, B.A.; *and others.*




Le point d'intersection S de MM′ et NN′ étant le pôle de AB par rapport au cercle (O), le lieu de S est la perpendiculaire abaissée de O sur AB. Mais le point R en question est le milieu du segment ST, où T est le point de rencontre fixe des perpendiculaires AT et BT sur AO et BO. Donc R parcourt la droite, qui bissecte perpendiculairement la droite AB.

**8951.** (W. J. C. Sharp.)—If $n$ be any whole number, which is not a multiple of 5, show that

(1) $x^{4n}+x^{3n}+x^{2n}+x^n+1$ is divisible by $x^4+x^3+x^2+x+1$,

(2) $x^{4n}+x^{3n}+x^{2n}+x^n+1$ ,, $x^4-x^3+x^2-x+1$, if $n$ be even,

(3) $x^{4n}-x^{3n}+x^{2n}-x^n+1$ ,, $x^4-x^3+x^2-x+1$, if $n$ be odd.

*Solution by* Professor Lampe.

Let $n$ and $p$ be relative prime numbers, then we have, identically,

$$\begin{aligned}(x^n)^p-1 \equiv (x^p)^n-1 \equiv x^{np}-1 &\equiv (x^{n(p-1)}+x^{n(p-2)}+\ldots+x^n+1)(x^n-1) \\ &\equiv (x^{p(n-1)}+x^{p(n-2)}+\ldots+x^p+1)(x^p-1),\end{aligned}$$

or dividing by $x-1$,

$$(x^{n(p-1)}+x^{n(p-2)}+\dots+x^n+1)(x^{n-1}+x^{n-2}+\dots+x+1)$$
$$\equiv (x^{p(n-1)}+x^{p(n-2)}+\dots+x^p+1)(x^{p-1}+x^{p-2}+\dots+x+1).$$

As no root of $\frac{x^n-1}{x-1}=0$ coincides with a root of $\frac{x^p-1}{x-1}=0$, $p$ and $n$ being relative prime numbers, all linear factors of $x^{p-1}+x^{p-2}+\dots+x+1$ must divide $x^{n(p-1)}+x^{n(p-2)}+\dots+x^n+1$, that is to say,

$x^{n(p-1)}+x^{n(p-2)}+\dots+x^n+1$ is divisible by $x^{p-1}+x^{p-2}+\dots+x+1$.

Similarly for the other two cases.

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**10092.** (H. W. Segar.)—Three ellipses A, B, C are similar, and similarly situated. A, B are also concentric. C cuts A in P, Q; and B in R, S. Show that PQ, RS intercept, on the line joining the centres of the conics, a constant length.

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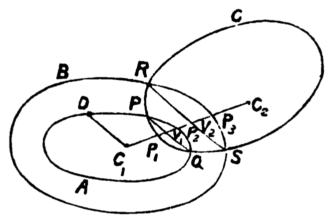
*Solution by* Rev. J. L. Kitchin, M.A.; H. W. Segar; *and others.*




Let $C_1$, $C_2$ be the centres of the conics. Let $C_1$, $C_2$ intersect the conics in points $P_1$, $P_2$, $P_3$, and the lines PQ, RS in $V_1$, $V_2$. Then PQ, RS are both conjugate to $C_1$, $C_2$. Let $C_1D$ be the semi-conjugate diameter to $C_1P_2$ in the ellipse A.

Then we have

$RV_2^2 : C_2P_1^2 - C_2V_2^2 = CD^2 : CP^2$,

and $PV_1^2 : C_2P_1^2 - C_2V_1^2 = CD^2 : CP^2$.

Hence $\quad RV_2^2 - PV_1^2 : C_2V_1^2 - C_2V_2^2 = CD^2 : CP^2$ ............... (1).

Again, $\quad RV_2^2 : C_1P_3^2 - C_1V_2^2 = CD^2 : CP^2$.

and $\quad PV_1^2 : C_1P_2^2 - C_1V_1^2 = CD^2 : CP^2$.

Hence $RV_2^2 - PV_1^2 : C_1V_1^2 - C_1V_2^2 + C_1P_3^2 - C_1P_2^2 = CD^2 : CP^2$ ...... (2).

Thus $\quad C_2V_1^2 - C_2V_2^2 = C_1V_1^2 - C_1V_2^2 + C_1P_3^2 - C_1P_2^2$.

This gives $\quad 2.V_1V_2.C_1C_2 = CP_3^2 - CP_2^2$.

All the ratios $CD^2 : CP^2$ should be $C_1D^2 : C_1P_2^2$.

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**10070.** (Professor Schoute.)—Find the locus of the centre of an equilateral triangle circumscribed to a given ellipse.

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*Solution by* W. S. Foster, M.A.; G. G. Storr, M.A.; *and others.*

Let $(x, y)$ be the centre of the equilateral triangle, $p$ the length of the perpendiculars from the centre on the sides $\phi$, $\phi+\frac{2}{3}\pi$, $\phi+\frac{4}{3}\pi$, the angles

which these lines make with the axis of $x$, then if the triangle circumscribes the ellipse $b^2x^2+a^2y^2 = a^2b^2$,

$$(x\cos\phi+y\sin\phi-p)^2 = a^2\cos^2\phi+b^2\sin^2\phi,$$

$$\{x\cos(\phi+\tfrac{2}{3}\pi)+y\sin(\phi+\tfrac{2}{3}\pi)-p\}^2 = a^2\cos^2(\phi+\tfrac{2}{3}\pi)+b^2\sin^2(\phi+\tfrac{2}{3}\pi),$$

$$\{x\cos(\phi+\tfrac{4}{3}\pi)+y\sin(\phi+\tfrac{4}{3}\pi)-p\}^2 = a^2\cos^2(\phi+\tfrac{4}{3}\pi)+b^2\sin^2(\phi+\tfrac{4}{3}\pi),$$

adding these, $\quad p^2 = \tfrac{1}{2}(a^2+b^2-x^2-y^2).$

Multiplying by $\cos\phi$, $\cos(\phi+\tfrac{2}{3}\pi)$, $\cos(\phi+\tfrac{4}{3}\pi)$ respectively, and adding,

$$(x^2-a^2-y^2+b^2)\cos 3\phi+2xy\sin 3\phi = 4px;$$

similarly, multiplying by sines, and adding,

$$(x^2-a^2-y^2+b^2)\sin 3\phi-2xy\cos 3\phi = 4py,$$

$$\therefore\ (x^2-y^2-a^2+b^2)^2+4x^2y^2 = 16p^2(x^2+y^2) = 8(x^2+y^2)(a^2+b^2-y^2-y^2),$$

or $$9(x^2+y^2)^2-8(a^2+b^2)(x^2+y^2)-2(x^2-y^2)(a^2-b^2)+(a^2-b^2)^2 = 0,$$

or $$\{3(x^2+y^2)-(a^2+b^2)\}^2 = 4(a^2x^2+b^2y^2+a^2b^2).$$

[This equation represents a circular quartic, which touches the given ellipse in the four points where it intersects the circle $x^2+y^2 = \tfrac{1}{3}(a^2+b^2)$.]

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**10260.** (Professor LEMAIRE.)—On donne un triangle ABC, inscrit dans un cercle de rayon B, et l'on joint deux à deux les milieux A′, B′, C′ des côtés BC, CA, AB. Si $\alpha$, $\beta$, $\gamma$ désignent les distances du centre O aux côtés du triangle ABC. et $\alpha'$, $\beta'$, $\gamma'$ les distances analogues relatives au triangle A′B′C′, on a $R^3 = \alpha^2\beta^2\gamma^2/\alpha'\beta'\gamma'$.

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*Solution by* Rev. T. R. TERRY, M.A.; J. C. HOROBIN, B.A.; *and others.*

If ON be the perpendicular on A′B′, then, since $\text{OAN} = \tfrac{1}{2}\pi-\text{B}$, we have $\alpha = \text{OA}' = \text{R}\cos\text{A}$, and $\gamma' = \text{ON} = \text{OA}'\sin\text{OAN} = \text{R}\cos\text{A}\cos\text{B}$, whence, &c.

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**10063.** (Professor MOREL.)—Dans un triangle ABC dont les côtés sont en progression arithmétique, démontrer que le plus grand angle A est lié au plus petit angle C par la relation

$$4(1-\cos\text{A})(1-\cos\text{C}) = \cos\text{A}+\cos\text{C}.$$

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*Solution by* R. KNOWLES, B.A.; J. J. BARNIVILLE; *and others.*

Let $a = c+2d$, $b = c+d$; then we have

$$1-\cos\text{A} = (c+3d)/2c;\quad 1-\cos\text{C} = (c-d)/2(c+2d),$$

and $$\cos\text{A}+\cos\text{C} = (a+c)(b+a-c)(b+c-a)/2abc$$
$$= (c-d)(c+3d)/c(c+2d);$$ whence the result.

**10169.** (R. H. W. Whapham, B.A.)—If the area of the triangle whose base is the polar of its vertex with respect to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is constant and equal to $k \,.\, ab$; prove that the locus of the vertex is $\quad (x^2/a^2 + y^2/b^2 - 1)^3 = k^2\,(x^2/a^2 + y^2/b^2)^2.$

*Solution by* W. J. Greenstreet, M.A.; W. H. Rees; *and others.*

If O be the pole $(l, m)$ and OP, OQ be the tangents, we know the

$$\Delta OPQ = (b^2l^2 + a^2m^2 - a^2b^2)^{\frac{3}{2}} / (b^2l^2 + a^2m^2) = k\,ab;$$

hence, for locus of $l$, $m$, we have

$$(b^2l^2 + a^2m^2 - a^2b^2)^3 = k^2a^2b^2\,(b^2l^2 + a^2m^2)^2;$$

thus the locus of pole is as stated in the Question.

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**10094.** (Capitaine de Rocquigny.)—La somme des $n^2$ termes des $n$ progressions arithmétiques 1, 3, 5 ..., 1, 5, 9 ..., 1, 7, 13 ..., &c., composées chacune de $n$ termes, est exprimée par un nombre triangulaire.

*Solution by* R. Knowles, B.A.; Rev. J. L. Kitchin; *and others.*

There are $n$ terms in each series, and thus $n^2$ terms in all; and the sums of the separate series, in order, are $n^2$, $n\,(2n-1)$, $n\,(3n-2)$, ... to $n$ terms; therefore

$$\begin{aligned} \Sigma &= n^2\,(1 + 2 + \ldots \text{ to } n \text{ terms}) - n\,(1 + 2 + \ldots \text{ to } n-1 \text{ terms}) \\ &= \tfrac{1}{2}n^2 . n\,(n+1) - \tfrac{1}{2}n^2\,(n-1) = \tfrac{1}{2}n^2\,(n^2+1), \text{ a triangular number.} \end{aligned}$$

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**9764.** (R. Knowles, B.A.)—A third tangent to an ellipse at a point R meets two tangents from a point T in MN; if O be the mid-point of MN, C the centre, R′ the end of the diameter through R; prove that CO is parallel to R′T.

*Solution by* Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

If T be $(h, k)$, and $\qquad y = mx + (a^2m^2 + b^2)^{\frac{1}{2}}$ ........................(1),

the two tangents to the ellipse through T are given by

$$(b^2h^2 + a^2k^2 - a^2b^2)(b^2x^2 + a^2y^2 - a^2b^2) = (b^2hx + a^2ky - a^2b^2)^2 \ldots\ldots(2).$$

If X = the abscissa of O, then we have

$$\begin{aligned} X &= \left[\left\{(a^2 - h^2)\,m + hk\right\}(a^2m^2 + b^2)^{\frac{1}{2}} - a^2km - b^2h\right] / \left\{(hm-k)^2 - (a^2m^2 + b^2)\right\} \\ &= \left\{a^2m - h\,(a^2m^2 + b^2)^{\frac{1}{2}}\right\} / \left\{(hm-k) - (a^2m^2 + b^2)^{\frac{1}{2}}\right\}. \end{aligned}$$

The equation of the straight line through C parallel to R′T is

$$\left\{h\,(a^2m^2 + b^2)^{\frac{1}{2}} - a^2m\right\} y = \left\{k\,(a^2m^2 + b^2)^{\frac{1}{2}} + b^2\right\} x \ldots\ldots\ldots\ldots(3),$$

and it may be shown that (3) meets (1) at the point of which the abscissa is X, that is, at O. Hence CO is parallel to R′T.

**10131 & 10167.** (H. L. Orchard, M.A., B.Sc.)—Find the relation that must subsist among the coefficients of the cubic $x^3 + ax^2 + bx + c = 0$ in order that it may be solvable by a simple quadratic method.

*Solution by* J. J. Barniville ; A. Gordon ; *and others.*

The equation $x^3+ax^2+bx+c=0$ can be depressed to a quadratic by factorising, if (i.) $a=c/b$, *i.e.*, $ab-c=0$; or, if (ii.) $c^{\frac{1}{3}}=b/a$, *i.e.*, $ac^3-b^3=0$. The condition is, therefore, $a^4bc-a^3c^2-ab^4+b^3c=0$, which, if $a=c=p$, $b=q$, becomes $(q-1)(p^4-q^3)=0$.

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**10219.** (Professor Sylvester.)—(1) Three rigidly connected closed curves (without singular points) in a plane are thrown down on a grating composed of an indefinite number of parallel lines, the distance between any two consecutive ones of which is uniform and not less than the greatest diameter of the complex contour formed by the three given curves. Show that, if certainty is represented by the periphery of a circle touched by two consecutive parallels, the probability of intersection between the complex contour and the parallels will be represented by some one of a determinate group of homogeneous linear functions, with (positive, negative, or zero) integer coefficients, of the lengths of the double tangents that can be drawn between the curves and of the segments into which the curves are divided by their points of contact with those tangents.

(2) Let there be any two rigidly connected figures whatever (A and C) in a plane; suppose an endless string to be passed round them, crossing itself at O, and let a third figure (B *intermediate* between A and C), rigidly connected with A and C, lie within either of the two open angles of the crossing string. Round A and C pass a tight uncrossed endless elastic band (which B is supposed large enough to intersect), and bend in the part of it which spans the angle in which B lies, towards O, until it passes round and rests on B. Show, on the same suppositions as previously made, that the probability of A, B, C being all simultaneously cut by one of the parallels, will be equal to the gain in length of the elastic band in passing from its first position to the second.

(3) If, everything else remaining the same as in (2), the figure B does not intersect the uncrossed elastic band round A and C, show that the probability of A, B, C being all cut by a parallel, is the difference in length between the bands obtained by making the part of this band which spans the open angle in which B lies, twist right round B in opposite directions.

In (2) it is to be understood that A, B, C are so situated that, of *any* straight line cutting them all three, the portion lying upon B will be intermediate between the portions lying upon A and C.

[For the corresponding, but very much simpler, theory of *two* figures, Professor Sylvester refers to Czuber's *Geometrische Wahrscheinlichkeiten*, Leipzig, 1884, pp. 117, 118, 125.]

I. *Solution by* D. Biddle.

The principle of the following solution is hereinafter extended to cases in which the three curves are at a distance from each other, and can readily be applied to any number of closed curves, whether separate or joined.

(1) Assuming, in the first instance, that the "complex contour" of the question is formed by closed curves which touch or overlap, so as to constitute a single figure, the function referred to is the sum of the three double tangents, represented by DE, FG, HK (Fig. 1), and of the three curve-segments, EF, GH, KD, extending between them so as to complete the circuit. Moreover, the same principle holds good for any figure; for *the ratio borne to the periphery of the given circle (touching two adjacent parallels) by the length of a string drawn tightly round any plane figure whatsoever (capable of lying in all positions between the parallels), is invariably the pro bability*

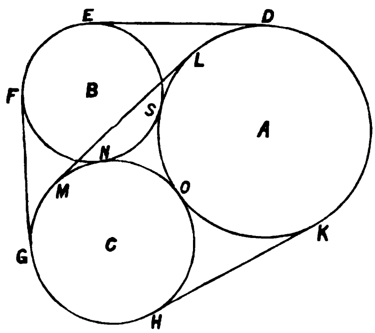


Fig. 1.

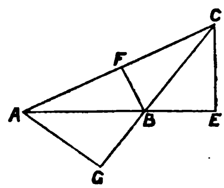


Fig. 2.

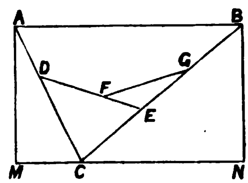


Fig. 3.

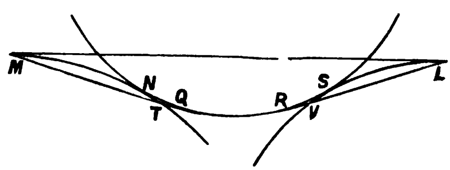


Fig. 4.

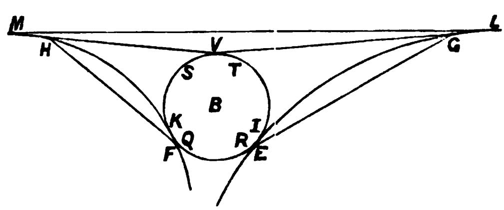


Fig. 5.

*that such figure will be cut by a parallel*, if disposed at random on a ruled floor. Thus, to begin at the figure (if such it can be called) which has only two prominent points, namely, the finite straight line; if its length equal the distance between two adjacent parallels, the probability of its cutting a parallel is $2/\pi$, or the ratio borne to the periphery of the given circle by the length of a string passing round the extreme points of the line. For a line of shorter length the probability is proportionally less. The equilateral triangle, the square, and all regular $n$-gons are readily seen to follow the same rule, and in their case

$$P = 2r\,\frac{n}{\pi}\int_0^{\pi/n} \cos\phi\,.\,d\phi = 2rn\sin\frac{\pi}{n}\Big/\pi,$$

where $r \equiv$ the radius of the circumscribing circle, in proportion to the distance between parallels. Again, if we take any triangle which is not equilateral, such as ABC (Fig. 2), we easily find that its probability

$$= (CF - BE) + (AF - BG) + (CG + AE),$$

divided by the periphery of the given circle, $= (AC + BC + AB)$ divided by the same. And from the triangle we can rise to any multilateral convex figure. Thus, if $Z \equiv$ the periphery of the given circle, touching parallels indefinitely wide apart, and if ABC (Fig. 3) be the triangle formed by producing two sides of the quadrilateral ADEB, then

$$(AB + BC + AC)/Z = \text{the probability for ABC};$$

and likewise, $(CD + DE + EC)/Z =$ the probability for CDE.

Now, if the triangle ABC revolve about its apex C, the probability that a line parallel to MN shall cut AC, BC, without cutting AB, is proportioned to the height above MN of the lowest point on AB; and the average representative of the probability

$$= AC - CM + BC - CN = AC + BC - AB.$$

Similarly, the probability of a line cutting CD, CE, without cutting DE, is proportioned to $CD + CE - DE$. Therefore the probability in regard to the quadrilateral, ADEB,

$$= \{(AC + BC + AB) - (CD + CE - DE)\}/Z = (AB + AD + DE + BE)/Z.$$

But, as we have used the triangle to prove the case for the quadrilateral, we can, by joining FG (Fig. 3), use the quadrilateral to prove the case for the pentagon, and so on, until the convex $n$-gon, regular or irregular, is lost in any curved or composite figure having a convex perimeter. It is further easy to see that indentations of any kind which do not affect the prominent points of a figure, or alter its contour as indicated by the imaginary string drawn tightly round it, do not affect the probability as to its intersection by a parallel.

(2) Let LM (Fig. 4) be the portion of elastic band spanning the interval between A and C, and MNQRSL its position when bent in round B, MN and LS being arcs of C and A respectively, as shown also in Fig. 1. Draw MQ, LR tangential to B. Then it is evident that, in order to cut A B C in the order named (or the reverse), the parallel must in-

tersect the figure LMQR, and not only so, but be subject to still greater restriction. The majority of intersections which fulfil the conditions avoid LM, but not all, for a parallel will sometimes intersect MNT or LSV without crossing MT, LV, and then it necessarily intersects LM near one or other extremity. Now, by reference to Fig. 3, and the kind of proof already given, the probability that a parallel shall cross CD, CG, without crossing DF, FG, is seen to be

$$\{(CD + CG) - (DF + FG)\} / Z,$$

and the same principle holds throughout. Join NT, SV (Fig. 4). Then the probability that a parallel shall intersect MN, NT, without crossing MT, is given by $(MN + NT - MT) / Z$, and on the other side we have $(LS + SV - LV) / Z$. But some of these intersections must be excluded, because the parallel may cross NT, TQ or SV, VR without intersecting NQ, SR respectively. Consequently, the probability in the question is given by

$$\{(MQ + QR + RL - LM) + (MN + NT - MT) + (LS + SV - LV)$$
$$-(NT + TQ - NQ) - (SV + VR - SR)\}/Z,$$

which reduces to $(MNQRSL - LM) / Z$, as stated.

(3) Let L, M (Fig. 5) be points similar to those so-named in Fig. 1. Let the double tangents GS, HT cross at V (which is not on the periphery of B). Also draw GR, HQ tangential to B. Then the two positions of the elastic band are defined by HKTSIG and HTRQSG, the latter crossing itself at V, the former between Q and R. Reference to the proof given under (2) shows that the probability in the present case is the difference between HKQRIG and HV + VG, less the difference between SV + VT and the arc ST; and this reduces to the difference in length of the elastic band in the two positions.

In the foregoing solution "grating" is treated as equivalent to "ruled floor," because in (2) and (3) of the question the results given are those for a figure lying flatly upon the plane, and not for one falling obliquely and possibly missing a parallel which it would otherwise strike. But since the probability for any figure under the conditions stated in (1) equals that of a circular disc having a periphery equal in length to the imaginary string drawn tightly round the figure, it is clear that the diameter of such circular disc represents the average breadth of the figure, as measured between parallel tangents during rotation about a perpendicular axis through its centroid. And if a real grating be substituted for a ruled floor, the same ratio will obtain between the probabilities in the two cases; so that, as the probability that a circular disc, of diameter equal to the distance between wires, shall in falling strike one of them, is $\frac{1}{4}\pi$, provided the normal to the disc point to any part of the heavens impartially, we need only multiply the proportionate length of the string, as compared with the periphery of the standard disc, by $\frac{1}{4}\pi$, in order to obtain the probability that a wire shall be struck by any figure which the string measures. Thus, taking a rod of infinitesimal thickness, but of length equal to the distance between wires, we have already seen that its probability of lying across a parallel on a ruled floor is $2/\pi$; consequently the probability of its striking a wire is $(2/\pi) \times \frac{1}{4}\pi = \frac{1}{2}$, which we know to be right, this being double the average distance of the points on

a sphere from a plane through its centre, when the diameter is unity. The probabilities of an equilateral triangle and of a square, whose circumscribing circles equal the standard disc, are $\frac{3}{8}\sqrt{3}$ and $\sqrt{\frac{1}{2}}$ respectively; and *the probability that any figure not exceeding the stated dimensions shall strike a wire, may always be found by dividing the length of the imaginary string drawn tightly round it, by four times the distance between wires,* where the grating is real.

As to the intersection of the three component parts of the figure, in a given order, by a vertical plane rising from one of the parallels, as suggested by (2) and (3) of the question, the same law is followed. For, if we suppose a disc having a circumference equal to the periphery of the special portion of Fig. 4 defined by MNQRSLM, and then mark off a segment of the circumference equal to LM, we find that lines drawn through the centre of the disc intersect the segment in the ratio (as to frequency) of twice its length to the whole circumference. Consequently, the difference between the segment and the remainder of the circumference represents the proportion of lines missing the said segment and yet crossing the disc through its centre. We have already found the same kind of thing to hold in regard to the side of a triangle or other figure, when the latter is rotated about its apex or its centroid on a ruled floor or other horizontal plane; and it is clear that obliquity produces on the average the same results in both cases, that of the disc and of the special figure, if impartially directed to all parts of the heavens. The difference between the two portions of the periphery, as compared with the whole, represents the average ratio borne by successful intersections of the special enclosure to the total number, when the figure is thrown on a ruled floor, where it lies flat; and this relative proportion is unaffected by obliquity, which, however, undoubtedly reduces the probability of intersection of all kinds. Accordingly, in the cases supposed under (2) and (3) of the question, the probability of the vertical plane intersecting the three portions of the figure in the given order, is found by dividing the stated difference by four times the distance between parallels.

The principle of the foregoing solution may be extended to cases in which the constituent curves are not in contact, but only fixed in particular relative positions in the same plane. Thus, to take two out of the numerous cases that can be conceived, let A, B, C (Fig. 6) be separated circles. Either of them may lie wholly on one side of a parallel

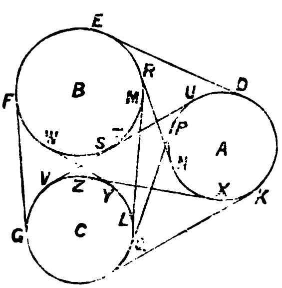


Fig. 6.

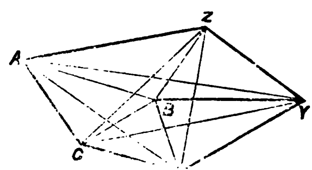


Fig. 7.

whilst the remaining two may lie on the other side. Consequently, from the length of the string DEFGHKD must be taken the sum of three differences, one of which is represented by (PQ + NR) − (PN + QLMR), in order to find the probability that at least one of the curves will be intersected. Again, let AZ, BY, CX (Fig. 7) be three thin ovals or needles in fixed relative position in the same plane, the required probability, under (1) of the Question, is found by subtracting from ACXYZA the sum of two differences,

$$(AY + CZ) - (AZ + CB + BY) \text{ and } (AX + CY) - (CX + AB + BY).$$

[Professor SYLVESTER remarks that Mr. BIDDLE has misapprehended the meaning of the Question, and has assumed the "figures rigidly connected," therein spoken of, to be in contact, in which case they are equivalent to the single figure obtained by passing a tight band round them. The proper solution will be found in a paper which is shortly to appear in the *Acta Mathematica*, in which all the distinguishable cases (11 in number) of three disjoined figures are severally considered, and a method given for extending the solution to four figures or more. Of these 11 cases 2 were singled out for solution in part (2) of the Question.]

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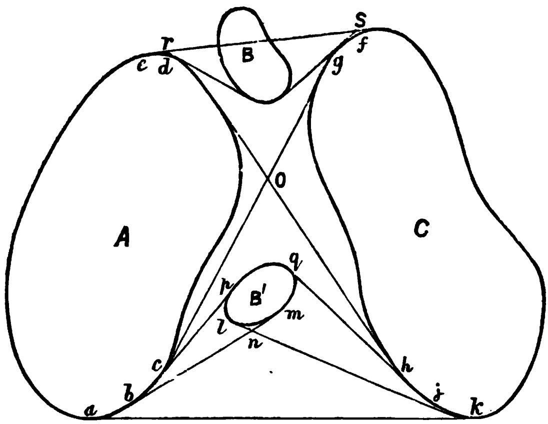
II. *Solution by* Professor SYAMADAS MUKHOPÂDHYÂY, M.A.

1. The chance of a random line intersecting a closed convex figure of perimeter $l$, is $l$, from the data of the question. The chance of a random




line intersecting two closed convex figures may be represented by the difference of the lengths of two bands going tightly around both, and forming two pairs of double tangents, and is therefore expressible as a function of lengths of arcs and double tangents of the nature indicated by the question. If the three closed curves without singularity be such that no line can intersect all three at the same time, then the chance of a random line intersecting the group is equal to the sum of the chances of

its intersecting the three curves, less by the sum of the chances of its intersecting the three pairs in which the curves can be taken. We may similarly resolve other cases, although some of them may require especial care. The chance, therefore, can always be represented by an expression of the kind indicated by the question.

2. From the accompanying figure, the chance of a random line intersecting A, B, C at the same time is the same as the chance of its intersecting *rdgs* at two points, that is, without cutting *rs*, minus the chances of its intersecting *erd*, *gsf* without cutting the arcs *ed*, *gf* respectively. The former chance is $rdgs - rs$, and the latter two chances are $rde - ed$ (*arc*) and $gsf - gf$. So that we have the chance of its intersecting A, B, C simultaneously equal to $edgf - ersf$.

3. In (2) it will be useful to see that the chance of a random line intersecting A, C, and B simultaneously is really the same as its intersecting the line *edgf* at *at least* two points. This latter chance is therefore equal to $edgf - ef$.

The chance of a random line intersecting A, B′, C simultaneously is the same as the chance of its intersecting the line *acpqhk* at *at least* two points minus the chances of its intersecting *abnjk* at *at least* two points and of its intersecting *lnm* without cutting the arc *lm*.

The actual chance is therefore

$$(acpqhk - ak) - (abnjk - ak) - (lnm - lm)$$
$$= acpqhk + lm + lpqm - abnjk - lnm - lpqm = acpqmlpqhk - abnmqplnjk.$$

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**10274.** (Rev. ROBERT HARLEY, M.A., F.R.S.)—"The members of a board were each of them either bondholders or shareholders, but not both; and the bondholders, as it happened, were all on the board. What conclusion can be drawn?" (VENN, *Mind*, vol. I., p. 487.) Show (1) that the conclusion, "No shareholders are bondholders," can be drawn from part of the premise, and (2) give *all* the conclusion.

I. *Solution by the* PROPOSER.

This question was originally proposed in a London University Examination, eighteen years ago, by Dr. VENN, who intended it to be answered by the aid of the symbolic method described by JEVONS in his *Lessons on Logic*. It affords a good illustration of that method. The proposer, in a paper on BOOLE's *Logical System*, mentions (*Mind*, vol. I., p. 487) that when it was first given in examination and lecture rooms, to some hundred and fifty students, as a problem in ordinary logic, it was answered by at most five or six of them; but that, when it was afterwards set as an example on BOOLE's method, to a small class who had attended a few lectures on symbolic logic, it was readily answered by half or more of their number. He adds in a footnote,—"The conclusion wanted is, 'No shareholders are bondholders.' Now nothing can look simpler than the following reasoning *when stated*:—'There can be no bondholders who are shareholders, for if there were they must be either on the board or off it. But they are not on it, by the first of the given statements; nor off it, by the second.' Yet, from want of any clue what to look for, almost every one, as above mentioned, failed to hit on so apparently obvious a solution." The little problem has been noted by JEVONS in his *Principles of Science*

(pp. 90–1, second edition), by MACFARLANE in his *Algebra of Logic* (pp. 140–1), and by others.

The object of this note is to point out that the question really consists of three statements, of which one is superfluous, so far as the "conclusion wanted" is concerned, and the other two are necessary and sufficient. The three statements are—

1. That all the members of the board, save directors, are either bondholders or shareholders.
2. That no director is both a bondholder and a shareholder.
3. That all the bondholders are directors.

Write $d$ for director, $b$ for bondholder, and $s$ for shareholder; then the given propositions may be expressed symbolically thus—

$$d = d(b+s-bs), \quad dbs = 0, \quad b = db \quad \text{..............} (1, 2, 3),$$

By substituting (3) in (2), we have at once $bs = 0$ ........................ (4).
which is the required solution, viz., *no bondholder is a shareholder.*

But if the question were—What is *all* the conclusion that can be drawn? we should need to combine (1) with (2) and (3), and follow the Boolian method. We should thus find, in addition to (4), the following relations, viz.,

$$b = d(1-s) \quad \text{..................................} (5),$$

or, *all the bondholders are all the directors who are not shareholders;* and

$$s = d(1-b) + 0/0\,(1-d) \text{..............................} (6),$$

or, *all the shareholders consist of all the directors who are not bondholders together with an indefinite number of persons who are not directors.*

In (4), (5), (6) we have *all* the conclusion which can be drawn from the given statements. JEVONS and MACFARLANE both unite the first and second statements in one equation.

[Dr. VENN remarks that he has often answered the question, when setting it to pupils, in the way suggested above, viz., by pointing out that the difficulty partly arises from the fact that the premise contains needless and superfluous information.]

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II. *Solution by* HUGH MACCOLL, B.A.

Putting $d$ for director (*i.e.*, one on the board), $b$ for bondholder, and $s$ for shareholder, our data are $d : (b+s)(bs)'$ and $b : d$. These two implications may be expressed as one zero implication, namely, $d(b's' + bs) + bd' : 0$. Reducing the antecedent to its *primitive form* (see my third paper in the *Proc. of the Lond. Math. Soc.*), we get $db's' + bs + bd' : 0$, the factor $d$ in the middle term disappearing. This asserts the impossibility of the terms in the antecedent separately and collectively. Solving first with regard to $b$, we get $ds' : b$ and $b : ds'$, two implications which are equivalent to the single equation $ds' = b$. These conclusions put into words are—*Every director who is not a shareholder is a bondholder;* and *every bondholder is a director but not a shareholder.* Solving next with regard to $s$, we get $db' : s$ and $s : b'$. These conclusions expressed in words are—*Every director who is not a bondholder is a shareholder; but no shareholder is a bondholder.* These two conclusions with regard to $s$ cannot (as in those with regard to $b$) be expressed as a single equation, since the antecedent of the one is not (as in the case of the $b$-conclusions) identical with the consequent of the other. There is no other conclusion which is not implied in those obtained.

**10270.** (The EDITOR.)—If the mid-points of the arcs cut off by the sides of a convex quadrilateral in a circle be joined so as to form a second quadrilateral, and a third be similarly formed from the second, and so on, find the ultimate form towards which these quadrilaterals tend.

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*Solution by* J. J. BARNIVILLE; Professor SCHOUTE; *and others.*

As the arithmetic mean of two quantities is greater than the less, and less than the greater of the two, the process is a nivellating one; therefore the limit of the quadrilateral is a square. Indeed, when the initial arcs are $\alpha, \beta, \gamma, \delta$, and $\epsilon$ represents successively $\alpha+\beta, \beta+\gamma, \gamma+\delta, \delta+\alpha$, the four arcs are, after the first, third, fifth, $2n-1$th divisions,

$$\tfrac{1}{2}\epsilon,\ \tfrac{1}{4}(\pi+\epsilon),\ \tfrac{1}{8}(3\pi+\epsilon),\ (1/2^n)\left\{(2^{n-1}-1)\pi+\epsilon\right\},$$

and the limit of $(1/2^n)\left\{(2^{n-1}-1)\pi+\epsilon\right\}$ for $n=\infty$ is $\frac{1}{2}\pi$, &c.

[The place of the limiting square cannot easily be assigned; we may, however, observe that there are two limiting squares, the vertices of which form the vertices of a regular octagon. The extension of the problem to inscribed polygons is evident.]

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**10296.** (Professor CROFTON, F.R.S.)—Prove that (1) the number of ways (N) in which $n$ things can be distributed among $x+r$ persons, in such a manner that *a particular set* of $r$ persons must each receive something, is $N = \Delta^r x^n$; and hence (2) the number of ways in which $n$ things can be distributed to $r$ persons, each receiving something, is $N = \Delta^r 0^n$. [If $n < r$, $N = 0$; if $n = r$, $N = n!$]

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*Solution by* Rev. T. R. TERRY, M.A.; F. R. J. HERVEY; *and others.*

For brevity, call a person who must receive something a restricted person, and let $(r, x)$ be the number of ways of distributing $n$ things (all different) among $r$ restricted and $x$ unrestricted persons; then, if $r=0$, clearly, we have $\qquad (0, x) = x^n$ ..........................................(1).

To find $(r, x)$, consider the effect of taking the restriction off a particular person A. Before removing the restriction there are $r$ restricted and $x$ unstricted persons, therefore $(r, x)$ distributions. Afterwards there are $(r-1)$ restricted and $(x+1)$ unrestricted, therefore $(r-1, x+1)$ distributions.

But the gain is clearly the number of distributions in which A gets nothing and the things are divided among the remaining $(r-1)$ restricted and $x$ unrestricted, *i.e.*, $(r-1, x)$ distributions; therefore

$(r-1, x+1) = (r, x) + (r-1, x)$ or $(r, x) = (r-1, x+1) - (r-1, x)$...(2);

therefore the expression $(r, x)$ is formed exactly in the same way as $\Delta_r x^n$, and by (1) these expressions are equal when $r=0$; therefore

$$N = (r, x) = \Delta^r x^n.$$

[Let $N \equiv f(x, r)$, and exclude a person A from the particular set; then

$f(x+1, r-1)$ is made up of $f(x, r)$ ways in which A receives something, and $f(x, r-1)$ ways in which he does not; or,

$$f(x, r) = f(x+1, r-1) - f(x, r-1) = \Delta f(x, r-1).$$

Hence $$f(x, r) = \Delta^r f(x, 0) = \Delta^r x^n = \Delta^r 0^n + x\Delta^{r+1} 0^n + \dots,$$

which agrees with the formula (2) on page 72 of Vol. XLV.]

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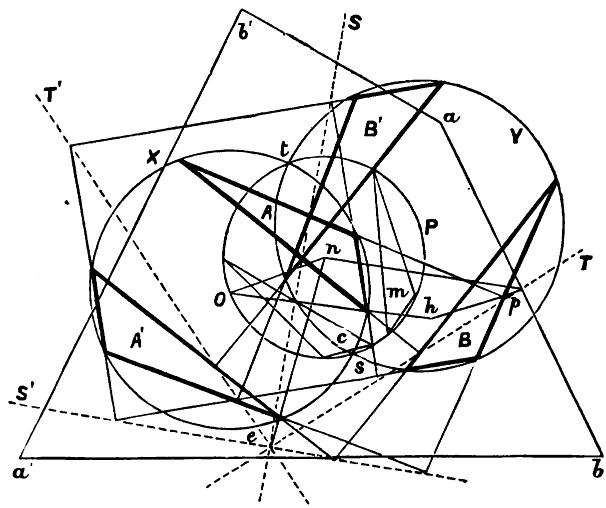
**10015.** (F. R. J. HERVEY.)—If B be the triangle formed by perpendiculars to the sides of a given triangle A at their intersections with any transversal T, prove that (1) the circumcircles (X, Y) of A, B are orthogonal; (2) the distance between their orthocentres is bisected by T; (3) to a given orthogonal circle Y correspond two transversals T, S, each of which, if the centre (P) of Y describe a circle of radius $k$ about that of X, envelopes a three-cusped hypocycloid the locus of whose centre, when $k$ varies, is the straight line which bisects at right angles the distance between the circumcentre and orthocentre of A; (4) if P describe any curve, the intersection of S, T describes an orthogonal projection of a similar curve.

*Solution by the* PROPOSER.




If circles X, Y (centres O, P) intersect at $s$, $t$, inscribed figures A, B in perspective at $t$ are similar; the same rotation, with expansion or contraction, about $s$ which turns X into Y, turning A into B. Conversely, in the case in question, if A, T, and hence B, are given, and if the line joining any pair of corresponding vertices $v$, $w$ of A, B meet X again at

$t$, the orthogonal circle through $t$, $w$ passes through the remaining vertices of B.

Let $a$, $b$ be the orthocentres of A, B, and let $\mu$ denote the ratio of O$a$ is radius of X. The mid-points of the segments $v$, $w$ form a triangle $m$ inscribed in the coaxal circle (centre $n$) through O, P; its orthocentre $p$ to the mid-point of $ab$, since the figure consisting of $s$, $m$, and $p$ is similar to $s$A$a$ and $s$B$b$.

Suppose A, B to rotate about their respective circumcentres with angular velocity $= 2$. The sides meeting at corresponding vertices $v$, $w$ form a quadrilateral concyclic with $s$, having $vw$ and T for diagonals or opposite sides; hence velocity of T $= 3$. When A, B have turned through two right angles, let them become A′, B′, T becoming T′, &c.; T, T′ are at right angles. The pairs of triangles A, B′ and A′, B are in perspective at $s$, and their axes S, S′ at right angles. The lines of these four triangles form three similar rectangles whose homologous sides are proportional to the perpendiculars from O on the sides of A; their centres, found by drawing parallels to their sides from O and P, form the triangle $c$, a *perversion* of $m$. It is evident that T, T′ meet at a point $e$ common to the circumcircles of the three rectangles; the distances of $e$ from the vertices of $c$ being in the above proportion, $e$ is the orthocentre of $c$. Similarly, S, S′ meet at $e$.

Let $m_1$, $c_1$, &c. be the vertices of $m$ and $c$ corresponding to $v_1$, $w_1$, &c. A, &c. rotating as before, the angular velocities of $m$, $c$ and the chords $m_1c_1$, &c. are $= 2$, 4, and 3, respectively. When $v_1$, $m_1$, $w_1$ coincide at $s$, the circles whose centres are $c_2$, $c_3$ pass through $s$; T becomes their common chord, and, being perpendicular to $c_2c_3$, passes through $c_1$. Hence T, is *always* parallel to the chords $mc$, and hence, by *symmetry*, passes through $p$. Similarly, S, &c. bisect the remaining sides of the quadrilateral $aba'b'$. The pairs TT′ and SS′ envelope two cardioids, which reduce to the point $n$ when A is equilateral.

With the motion hitherto supposed, compound a rotation, $= -2$, of the whole figure about O. A is now fixed; the angular velocities of O$n$, $np$, $ne$, and T are $= -2$, 0, 2, 1, respectively. Hence, completing the parallelogram O$nph$, T envelopes a 3-cusped hypocycloid whose vertices lie on the circle described by $p$ about the fixed point $h$ (or perpendicular bisector of O$a$, since it is easily seen that vector O$h = \frac{1}{4}(a'a + b'b)$. Length O$h = \frac{1}{2}\mu k$). The point $e$ describes an ellipse, the directions of whose axes are independent of $k$. If $x$, $y$ be the coordinates of P, referred to these axes, those of $e$ are $\frac{1}{2}(1+\mu)\,x$ and $\frac{1}{2}(1-\mu)\,y$, for *all* positions of P.

Taking V on axis of $x$, let the angles made with OV by O$a$, OP, O$v$, &c., and perpendiculars to T, S, be denoted by $\alpha$, $\varpi$, $\theta_1$, &c., $\tau$, $\sigma$, respectively, and that of $np$ with O$a$ ($=$ O$sn = \frac{1}{2}t$O$s$) by $\delta$;

$$\tau = \tfrac{1}{2}(\alpha + \delta - \varpi) = \tfrac{1}{2}t\text{O}a, \quad \sigma = \tfrac{1}{2}(\alpha - \delta - \varpi) = \tfrac{1}{2}s\text{O}a.$$

The directions of radii $hp$ to vertices of hypocycloid are given by $\varpi = \tau$, or $3\varpi = \alpha + \delta = \text{VO}h$. When $t$ is diametrically opposite to a vertex of A, T coincides with opposite side. (Hence, the variable hypocycloid always touches the lines of A.) Thus, when $\varpi = \theta_1 + \pi + \delta$, we have

$$\alpha + \delta - \varpi = \theta_2 + \theta_3, \text{ whence } \alpha = \theta_1 + \theta_2 + \theta_3 + \pi.$$

Or, if $\theta_1'$, &c. be the corresponding angles, measured from O$a$, we have OV determined by $-\alpha = \frac{1}{4}(\theta_1' + \theta_2' + \theta_3' + \pi)$.

It is evident that, from 1, 2, 3, of the Question, we may infer, as a limiting case, the corresponding properties of Simson's line.

**10323.** (A. W. PANTON, M.A.)—If the general equation of a circular cubic be $(x\cos\alpha+y\sin\alpha)(x^2+y^2)+ax^2+2hxy+by^2+2gx+2fy+c=0$, prove that (1) the coordinates of the double focus are

$$\tfrac{1}{2}(b-a)\cos\alpha-h\sin\alpha,\quad \tfrac{1}{2}(a-b)\sin\alpha-h\cos\alpha;$$

and (2) if the double focus of a nodal circular cubic be situated on the curve, the tangents at the node are rectangular.

*Solution by* Professor SEBASTIAN SIRCOM, M.A.

1. The real asymptote is parallel to $x\cos\alpha+y\sin\alpha=0$, and is found to be $\quad L\equiv x\cos\alpha+y\sin\alpha+a\sin^2\alpha-2h\sin\alpha\cos\alpha+b\cos^2\alpha=0$;

thus the equation may be written

$$L\left\{x^2+y^2-2x\left[\tfrac{1}{2}(b-a)\cos\alpha-h\sin\alpha\right]\right.$$
$$\left.-2y\left[\tfrac{1}{2}(a-b)\sin\alpha-h\cos\alpha\right]\right\}+k^2M=0,$$

whence the coordinates of the focus are those given in the question.

2. Taking the origin at the double focus on the curve, the equation may be written $\qquad u\equiv(x^2+y^2)(ax+by+c)+k^2x=0$;

hence, eliminating $k^2$ and $x^2+y^2$ from this and $du/dx=0$, $du/dy=0$, we obtain at the double point $2ax+2by+c=0$; but this is equivalent to $d^2u/dx^2+d^2u/dy^2=0$, which is the condition that the tangents at the double point shall be at right angles.

[We may observe that (2) may be proved by direct substitution; for when the expressions given for the coordinates of the double forces are substituted for $x$ and $y$ in

$$(x\cos\alpha+y\cos\alpha)(x^2+y^2)+ax^2+2hxy+by^2,$$

this reduces to the remarkably simple form $\frac{1}{8}(a+b)\left\{(a-b)^2+4h^2\right\}$.]

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**9139 & 9190.** (J. BRILL, M.A.)—9139. A set of $m$ points is taken on a parabola, having the point P for their centroid, and a second set containing $n$ points is also taken, having the point Q for their centroid. Prove that the tangents at the extremities of the diameters through P and Q meet at the centroid of the $mn$ points of intersection of the tangents at the $m$ points with those at the $n$ points.

9190. Three sets of points are taken on a parabola, the first containing $l$ points, the second $m$ points, and the third $n$ points. P is the centroid of the $l$ points, Q that of the $m$ points, and R that of the $n$ points, and O is the centre of the circumscribing circle of the triangle formed by the tangents at the extremities of the diameters through P, Q, R. Prove that O is the centroid of the centres of the circumscribing circles of the $lmn$ triangles that can be formed by taking a tangent at one of the $l$ points for one side, a tangent at one of the $m$ points for a second side, and a tangent at one of the $n$ points for a third side.

*Solution by* Professor SCHOUTE.

Though both problems are easily verified directly we prefer to give a geometrical demonstration based on elementary properties of the parabola.

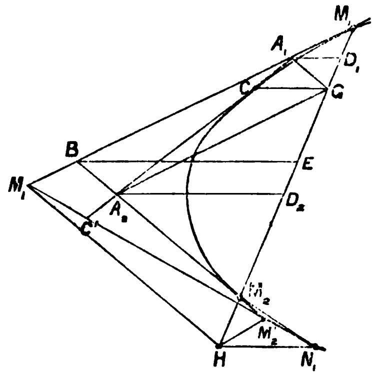


Fig. 1.

9139. In the first diagram $M_1$ and $M_2$ are two points of the set of $m$ points, and B is the pole of $M_1M_2$. Now the parallelogram $BA_1GA_2$ described on the tangents $BM_1$ and $BM_2$, the free vertex G of which is any point of $M_1M_2$, determines a third tangent $A_1A_2$ of the parabola, and its point of contact C lies on the diameter of G. For, as

$$BA_1 : A_1M_1 = M_2A_2 : A_2B,$$

the line $A_1A_2$ touches the parabola; this curve, being the envelope of the join of corresponding points of two similar divisions and the join of the point of contact C with G, is a diameter of the parabola, as we have

$$A_1C : CA_2 = M_1A_1 : A_1B$$
$$= M_1G : GM_2 = D_1G : GD_2.$$

Let $N_1$ be a point of the set of $n$ points, and $M_1'$, $C'$, $M_2'$ the points where the tangent in $N_1$ is met by the tangent in $M_1$, C, $M_2$. Then

$$M_1G : GM_2 = M_1'C' : C'M_2'$$

proves that $C'$ will be the centroid of the masses $\mu_1$ and $\mu_2$ placed in $M_1'$ and $M_2'$ when G is the centroid of those masses placed in $M_1$ and $M_2$. This proves that the centroid of the masses equal to unity, placed in $M_1'$, $M_2'$, ... $M_m'$, where the tangent in $N_1$ is met by the tangents in $M_1$, $M_2$, ... $M_m$, lies on the tangent in the point $P'$ of the parabola that belongs to the diameter of the centroid P of the points $M_1$, $M_2$, ... $M_m$. And when we now compound the $n$ masses $m$ placed in the points where the tangent in $P'$ meets the $n$ tangents in $N_1$, $N_2$, ... $N_n$, we find in the same manner that the centroid of the $mn$ equal masses is the point where the tangent in P meets the tangent in the point Q situated on the diameter of the centroid Q of the $n$ points $N_1$, $N_2$, ... $N_n$.

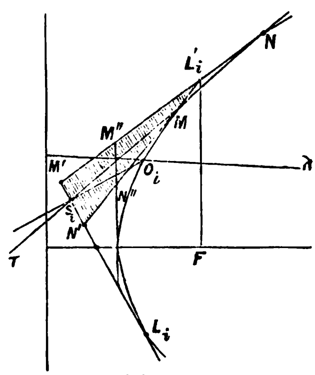


Fig. 2.

9190. In the second diagram $L_i$, M, N are three points belonging to the three different sets of $l$, $m$, $n$ points, and $L_i'M'N'$ is one of the $lmn$ triangles. Now we first consider the $l$ triangles obtained by substituting

for $L_i$ the $l$ different points $L_1, L_2, \ldots L_l$. Then of the triangle $L_i'M'N'$ only the side $M'N'$ changes. And as the circle circumscribed to $L_i'M'N'$ contains the focus F, the centres of these $l$ circles are situated on the line $\lambda$ that bisects $FL_i'$ orthogonally. Moreover, it is easily shown, that the centroid of this $l$ centres is the centre of the circle circumscribed to the triangle $L_i'\overline{MN}$, where $\overline{MN}$ indicates the tangent to the parabola in the point P′, the diameter of which passes through the centroid P of the $l$ points L. Indeed, the mid-points $S_i$ of the segments $M_i'N_i'$ lie on another tangent $\tau$ of the parabola, and the normals to these segments in the points of intersection with $\tau$ envelope another parabola, that touches $\tau$ and $\lambda$. For it is evident that these normals are the joins of the corresponding points of two homographic divisions on $\tau$ and the line in infinity, and $\lambda$ is one of the tangents as it bisects orthogonally the segment $M''N''$ on the tangent parallel to $L_i'F$ ($M''$, $N''$, F, $L_i'$ being four concyclic points). This new parabola proves, that the $l$ centres $O_i$ of the circumcircles form on $\lambda$ a range of points similar to the range of points $S_i$ on $\tau$, the corresponding points being situated on tangents to this new parabola. Therefore the centroid O of the $l$ points $O_i$ and the centroid S of the $l$ points $S_i$ are joined by a tangent of this new parabola; in other words, the centroid O is the centre of the circle circumscribed to the triangle formed by the tangents in M, N, and P′. For, according to problem 9139, the tangent $\overline{MN}$ in P′ passes through the centroid S of the points $S_i$, where $\tau$ meets the tangents in the $l$ points $L_1, L_2, \ldots L_l$, &c.

By what is shown, the composition of the $lmn$ masses equal to unity is reduced to the composition of $mn$ masses $l$ placed in the centres of the circles circumscribed to the $mn$ triangles formed by the tangent in P′, the tangent in one of the points M and the tangent in one of the points N. By a twofold repetition of this reduction the theorem is proved.

Remarks.—I. The centroid of the centroids of the $lmn$ triangles L′M′N′ is the centroid of the triangle formed by the tangents in P′, Q′, R′. This can be proved as follows. The centroid of the centroids of the triangles L′M′N′ is the centroid of the $3lmn$ vertices. These $3lmn$ points consist of $mn'$ points L′ counted $l$ times, $nl$ points M′ counted $m$ times, and $lm$ points N counted $n$ times. Now the centroid of the points L′ is P′, that of the points M′ is Q′, and that of the points N′ is R′. And these points P′, Q′, R′ are to be charged equally, *e.g.*, with $lmn$ unities of mass, when every vertex is loaded with the unity of mass. So the researched point is the centroid of triangle P′Q′R′.

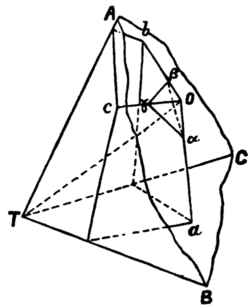
II. The centroid of the orthocentres of the $lmn$ triangles L′M′N′ is the orthocentrum of the triangle P′Q′R′. For when we place in the centroid and in the circumcentre of every triangle L′M′N′ respectively, the masses $+3$ and $-2$, we have placed the mass $-1$ in the orthocentre of every triangle L′M′N′. And now we can compound the masses $+3$ to a mass $+3lmn$ in the centroid of P′Q′R′, the masses $-2$ to a mass $-2lmn$ in the circumcentre of P′Q′R′, and finally the masses $+3lmn$ and $-2lmn$ to a mass $lmn$ in the orthocentre of P′Q′R′.

We must bear in mind that the orthocentres lie on the directrix.

III. The position of the centre of the nine-point circle on the line of Euler proves that the centroid of the nine-point circle-centre of the $lmn$ triangles L′M′N′ is the nine-point circle-centre of the triangle P′Q′R′.

**8943.** (W. J. C. SHARP, M.A.)—The angle between the great circles bisecting two angles of a spherical triangle, which is subtended by the third side, is the supplement of the angle contained by the chords of the corresponding arcs of the polar triangle.

*Solution by* Professor SCHOUTE.




Let T be the centre of the sphere, T (ABC) the solid angle corresponding to the given spherical triangle, and O any point of the line TO passing through the incircle of the spherical triangle. Let O$a$, O$b$, O$c$ be perpendicular to the planes TBC, TCA, TAB, and on these lines be taken O$\alpha$ = O$\beta$ = O$\gamma$. Then $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ are parallel to the chords of the polar triangle in question, and it is evident that these lines are respectively perpendicular to the planes TAO, TBO, TCO. And as the angle between two planes equals that formed by two perpendiculars on these planes, the theorem is proved.

[Mr. SHARP's Solution, not nearly so elegant as Professor SCHOUTE's, was obtained by solving the triangle OBC (O being the incentre) so as to find BOC, and comparing the result with the known angle between the chords. See TODHUNTER's *Spherical Trigonometry*, p. 73.]

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**10032.** (The EDITOR.)—Show that the values of $x$, $y$, $z$, from the equations

$$a(x-y+z)^{\frac12}(x+y-z)^{\frac12} = xy^{\frac12}z^{\frac12},\quad b(x+y-z)^{\frac12}(-x+y+z)^{\frac12} = yx^{\frac12}z^{\frac12},$$
$$c(-x+y+z)^{\frac12}(x-y+z)^{\frac12} = zx^{\frac12}y^{\frac12},$$

are
$$\frac{a}{bc}(b^2+c^2-a^2),\quad \frac{b}{ca}(c^2+a^2-b^2),\quad \frac{c}{ab}(a^2+b^2-c^2).$$

*Solution by* D. J. GRIFFITHS; C. MORGAN, M.A.; *and others.*

By multiplication, $$(x-y+z)^{\frac12}(x+y-z)^{\frac12}(-x+y+z)^{\frac12} = \frac{\pm xyz}{(abc)^{\frac12}}.$$

Dividing the corresponding members of this equation by the corresponding members of each of given equations, and squaring, we have

$$-x+y+z = \frac{ayz}{bc},\quad x-y+z = \frac{bzx}{ca},\quad x+y-z = \frac{cxy}{ab}.$$

Adding the first two equations, we obtain

$$2z = z\left(\frac{ay}{bc}+\frac{bx}{ca}\right);$$

therefore $$z = 0 \text{ or } a^2y+b^2x = 2abc.$$

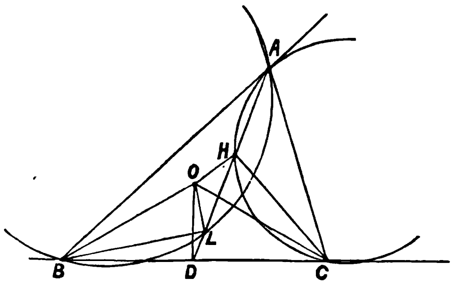
Similarly, $x = 0$ or $b^2z+c^2y = 2abc$, $y = 0$ or $c^2x+a^2z = 2abc$, whence follows the stated result.

**10281.** (E. M. LANGLEY, M.A.)—Circles are described to touch AB, AC, at A, and pass through C and B respectively. If the median through A meets these circles again in H and L, show that these points are equidistant from the circumcentre of ABC. Show also that BL = CH.

*Solution by* G. G. STORR, M.A.; D. T. GRIFFTHS, M.A.; *and others.*

We have $\angle BLD = 180° - BLA = BAC$;

similarly, $\angle CHD = BAC$; $\therefore \angle BLD = BOD$;



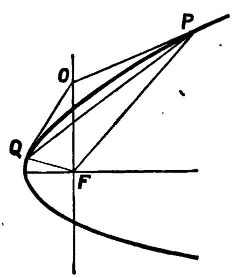

hence a circle will go round BOLD; $\therefore \angle OLH = OBD = 90° - BAC$;

similarly, $\angle OHL = 90° - BAC$; $\therefore OH = OL$.

Also $\angle OLB = 90° = OHC$ and $OB = OC$; $\therefore BL = CH$.

**10215.** (R. KNOWLES, B.A.)—From a variable point O on the latus rectum of a parabola, whose focus is F, tangents are drawn to meet the curve in P, Q; prove that the locus of the centre of the circle PQF is a semi-cubical parabola.

*Solution by* Rev. J. L. KITCHIN; Rev. T. GALLIERS; *and others.*




Take $FO = k_1$ — a variable; then the equation to PQ is $ky = 2a(a+x)$; and if $(\alpha, \beta)$ be the coordinates of the centre of the circle round PFQ, we easily obtain the relations

$$k^3 - (k^2 - 4a^2)^{\frac{1}{2}} \left\{k^2 + 4a(a-\alpha)\right\} - 8a^2\beta = 0,$$

$$k^3 + (k^2 - 4a^2)^{\frac{1}{2}} \left\{k^2 + 4a(a-\alpha)\right\} - 8a^2\beta = 0;$$

whence $$k^3 = 8a^2\beta,$$

$$ak^2 + 4a(a-\alpha) = 0;$$

thus $$(\alpha - a)^3 = a\beta^2,$$

is the locus of centres, a semi-cubical parabola.

**10199.** (D. T. GRIFFITHS.)—If circles can both be drawn round and in a quadrilateral whose sides are $a$, $b$, $c$, $d$, and the angle between the diagonals $\theta$, prove that, if $\theta < \frac{1}{2}\pi$,

$$\left(\frac{bd}{ac}\right) \theta = 2 - \tfrac{2}{3}\left(\frac{ac}{bd}\right) + \tfrac{2}{5}\left(\frac{ac}{bd}\right)^2 - \tfrac{2}{7}\left(\frac{ac}{bd}\right)^3 + \&c.$$

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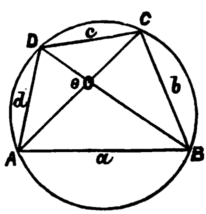
*Solution by* W. J. GREENSTREET; Rev. J. L. KITCHIN; *and others.*




If $A + C = B + D = \pi$, and $a + c = b + d$, the area,

quadrilateral $= \frac{1}{2}(ac + bd)\sin\theta = (abcd)^{\frac{1}{2}}$,

$\therefore$ $\tan\theta = 2\left(\frac{ac}{bd}\right)^{\frac{1}{2}} \Big/ \left(\frac{ac}{bd} - 1\right)$;

hence, expanding $\theta$ in terms of $\tan\theta$, and arranging coefficients, we get the stated result.

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**10202.** (MAURICE D'OCAGNE.)—Si les cordes PQ, PR d'une conique passent par les foyers de cette conique, démontrer que la normale à la conique en P et ses tangentes en Q et en R sont concourantes.

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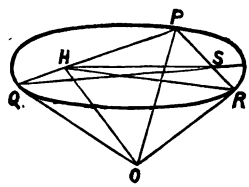
*Solution by* E. M. LANGLEY, M.A.; J. C. ST. CLAIR; *and others.*




Let the tangents at Q, R meet in O; then $\angle$ OHR = OHQ, *i.e.*, OH is the external bisector of $\angle$ PHR. Also OR is external bisector of $\angle$ PRH; therefore OP is internal bisector of $\angle$ P. Therefore OP is normal at P.

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**10291.** (L. W. ROBINSON, B.A.)—If R be radius of circumcircle of a triangle, and $\partial$, $\partial_1$, $\partial_2$, $\partial_3$ the distances of its centre from escribed centres, prove that $\partial^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 = 12R^2$.

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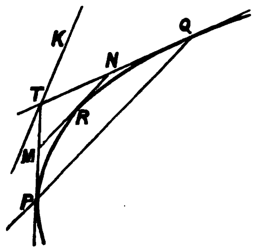
*Solution by* Rev. T. R. TERRY, M.A.; J. C. HOROBIN, M.A.; *and others.*

It is well known that $\partial^2 = R^2 - 2Rr$, and $\partial_1^2 = R^2 + 2Rr_1$, &c.; whence, adding and remembering that $r_1 + r_2 + r_3 - r = 4R$, we get the result.

**10248.** (R. Knowles, B.A.)—Tangents TP, TQ meet a parabola in P, Q; the diameter through T meets the curve in R; the tangent at R meets TP, TQ in M, N, respectively: prove that the circles TPQ, TMN touch at the point T.

*Solution by* G. G. Morrice, M.B.; R. Whapham, B.A.; *and others.*




Through T draw a tangent TK to the circle TPQ; therefore $\angle$ KTQ = $\angle$ TPQ. Now we know that MN is parallel to PQ;

therefore $\quad \angle$ TMN = $\angle$ TPQ,

therefore $\quad \angle$ KTQ = $\angle$ TMN,

therefore TK is a tangent to the circle TMN, at T; therefore, &c.

[As M and N are the mid-points of the segments TP and TQ, the circle TPQ is obtained by the multiplication of the radii vectores of the circle TMN by two, T being the origin of radii vectores; therefore, &c.]

**8815.** (Asparagus.)—The circle of curvature at a point P of a given ellipse meets the ellipse again in the point Q; prove that (1) the maximum angle between the two curves at Q (measured between the tangents drawn to the two at Q outside the ellipse) is $4\tan^{-1}(b/a)$, PQ being then one of the equal conjugate diameters of the ellipse; (2) if QQ′ be a chord of the circle touching the ellipse in Q, and $a\cos\theta$, $b\sin\theta$ the point P,

$$QQ' = 2(a^2-b^2)\sin^3 2\theta \,/\, (a^2\sin^2 3\theta + b^2\cos^2 3\theta).$$

*Solution by* Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

Let $(\alpha, \beta)$ be the centre of curvature at P, $\psi$, $\psi'$ the inclinations to the major axis of normals at P to the ellipse and circle of curvature respectively, then $\alpha = c^2\cos^3\theta/a$, $\beta = -c^2\sin^3\theta/b$, where $c^2 = a^2-b^2$. Let Q be $(x_1, y_1)$, Q′ $(x_2, y_2)$, then $x_1 = a\cos 3\theta$, $y_1 = -b\sin 3\theta$, also

$$\tan\psi = a^2y_1/(b^2x_1), \quad \tan\psi' = (y_1-\beta)/(x_1-\alpha).$$

Let $\psi'-\psi = \delta$, $\tan\delta = u$; then $u = 2ab\,.\,c^2\sin^3 2\theta\,/\,D$,

where $\quad D = 2a^4\sin 3\theta\sin^3\theta - 3a^2b^2\sin^2 2\theta - 2b^4\cos 3\theta\cos^3\theta.$

Hence $\quad du/d\theta = Kn\,/D^2,$

where $\quad n = \sin^2 2\theta\cos 2\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)^2,$

and K is independent of $\theta$. Now $u$ will be a maximum or minimum when $\theta = 0$, $\frac14\pi$, $\frac12\pi$, or $\frac34\pi$; $\theta = 0$ and $\theta = \frac12\pi$ give the extremities of the axes; and, as things repeat themselves in each quadrant, we need only consider the case of $\theta = \frac14\pi$. In this case

$$u = 4ab\,(a^2-b^2)/(a^4-6a^2b^2+b^4) = \tan 4\lambda, \text{ where } \tan\lambda = b/a.$$

The sign of $d^2u/d\theta^2$ will be found to depend on that of $-\sin^3 2\theta$, and is therefore negative when $\theta = \frac14\pi$, so that $u$ will be a maximum. When

$\theta = \frac{1}{4}\pi$, the equation of PQ is $bx - ay = 0$; thus PQ is one of the equi-conjugates.

(2) If $\chi$ = the inclination of QQ′ to the major axis, we may show that

$$\sec\chi = D'/(a\sin 3\theta), \quad \text{where } D'^2 = a^2\sin^2 3\theta + b^2\cos^2 3\theta;$$

and by writing down the equations of QQ′, and of the circle of curvature at P, and combining them, we have

$$x_2 - x_1 = 2ac^2\sin 3\theta\sin^3 2\theta / D'^2.$$

Hence $$QQ' = (x_2 - x_1)\sec\chi = 2c^2\sin^3 2\theta / D',$$

the required result.

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**9179.** (R. Knowles, B.A.)—A circle of curvature is drawn at a point $P(m, a^2/m)$ of the rectangular hyperbola $xy = a^2$; PQ is their common chord, and R the point of contact of their common tangent with the hyperbola; show that

$$\Delta RPQ = \left\{2(a^4 - m^4)^5(a^4 + m^4)\right\} / \left\{a^2m^4(3m^8 + 6a^4m^4 - a^8)(3a^8 + 6a^4m^4 - m^8)\right\}.$$

*Solution by* Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

In the equation to the circle of curvature at P,

$$a^2m^3(x^2 + y^2) - a^2(a^4 + 3m^4)x - m^2(3a^4 + m^4)y + 3a^2m(a^4 + m^4) = 0,$$

put $y = a^2/x$, and the result $= (x - m)^3(m^3x - a^4)$; therefore the coordinates of Q are $a^4/m^3$, $m^3/a^2$, and the length of PQ

$$= (a^4 - m^4)(a^4 + m^4)^{\frac{1}{2}} / a^2m^3 \quad \ldots\ldots\ldots\ldots(1),$$

and its equation $$m^3x + a^2my = a^4 + m^4 \quad \ldots\ldots\ldots\ldots(2).$$

The length of the perpendicular from the centre of curvature on the tangent at R = radius of curvature; whence we find the coordinates of R,

$$m(3a^8 + 6a^4m^4 - m^8)/(3m^8 + 6a^4m^4 - a^8),$$
$$a^2(3m^8 + 6a^4m^4 - a^8)/m(3a^8 + 6a^4m^4 - m^8);$$

we can now find the perpendicular distance from R on PQ, and thence the area of $\Delta$PQR, the result being as in the question.

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**10105.** (Professor Mathews, M.A.)—Prove that (1) if $i^2 = -1$,

$$(7 + 2i)^{\frac{1}{2}} = 3 + \frac{1}{(-1-2i)+}\,\frac{1}{(-1-i)+}\,\frac{1}{2i+}\,\frac{1}{(1-2i)+}\,\frac{1}{(1-i)+}\,\frac{1}{5+}\ldots,$$

where all the quotients except the first recur;

and (2) $$(17 - 63i)^2 - (7 + 2i)(3 - 24i)^2 = 1.$$

*Solution by* G. E. Crawford, B.A.

1. Let $$x \equiv \frac{1}{(-1-2i)+}\,\frac{1}{(-1-i)+}\,\frac{1}{2i+}\,\frac{1}{(1-2i)+}\,\frac{1}{1-i+}\,\frac{1}{5+}\ldots$$

Now $$\frac{1}{1-i+\dfrac{1}{5+x}} = \frac{5+x}{6+x-i\,(5+x)} = a, \text{ say,}$$

$$\frac{1}{(1-2i)+a} = \frac{6+x-i\,(5+x)}{1-i\,(17+3x)} = \beta, \quad \frac{1}{2i+\beta} = \frac{1-i\,(17+3x)}{40+7x-i\,(3+x)} = \gamma,$$

[The rest in vol.]

$$\frac{1}{(-1-i)+\gamma} = \frac{40+7x-i\,(3+x)}{-42-8x-i\,(54+9x)} = \delta,$$

$$\frac{1}{(-1-2i)+\delta} = \frac{-42-8x-i\,(54+9x)}{-26-3x+i\,(135+24x)} = x;$$

hence $$(24i-3)\,x^2+x\,(144i-18)+42+54i = 0,$$

therefore $$x = -3 \pm (7+2i)^{\frac{1}{2}},$$

therefore $$(7+2i)^{\frac{1}{2}} = 3+x = 3+\frac{1}{(-1-2i),}\ \&c.$$

2. $(17-63i)^2-1 = (16-63i)(18-63i) = 9\,(1-8i)(1-8i)(7+2i)$
$= (7+2i)(3-24i)^2,$

therefore $$(17-63i)^2-(7+2i)(3-24i)^2 = 1.$$

[This depends on property that the product
$$(a-bi)(c-di) \equiv (b+ai)(d+ci).]$$

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**9947.** (The Editor.)—The ordinate of a point in a conic measured from the axis, is produced till the whole line bears a given ratio to the focal distance of the point; show that the locus of the end of the line is a straight line.

*Solution by* R. F. Davis, M.A.; R. Knowles, B.A.; *and others.*

Let the ordinate NP be produced to Q, so that QN : SP is constant. Then QN : NX is also constant, where X is the foot of the S-directrix. The locus of Q is therefore a fixed straight line through X.

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**10074.** (Professor Gob.)—Le centre radical des sommets B, C d'un triangle ABC et du cercle $(N_a)$ de Neuberg est le sommet $A_1$ du premier triangle de Brocard.

*Solution by* Professor Nash; G. G. Storr, M.A.; *and others.*

The equation of the circle is $SL-C = 0$, where $S = a\alpha+b\beta+c\gamma = 2\Delta$, $L = g\,(b\beta+c\gamma)/(bc)$, $C = a\beta\gamma+b\gamma\alpha+c\alpha\beta$. Hence square of tangent from $P_1\,(\alpha\beta\gamma)$ to the circle $= T^2 = abc\,(SL-C)/4\Delta^2$. But $PB^2 = PC^2 = T^2$;

hence, dividing by $abc/4\Delta^2$,

$$2\Delta\,(c\alpha+a\gamma)/b-C = 2\Delta\,(a\beta+b\alpha)/c-C = SL-C = 2\Delta g\,(b\beta+c\gamma)/bc,$$

therefore $$a\,(b\beta+c\gamma) = c\,(c\alpha+a\gamma) = b\,(a\beta+b\alpha),$$

or $$\alpha/abc = \beta/c^3 = \gamma/b^3.$$

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**9965.** (S. Tebay, B.A.)—Find positive integral values of $a_1, a_2, a_3, a_4$ such that $a_1a_2+a_3a_4$, $a_1a_3+a_2a_4$, $a_1a_4+a_2a_3$, and

$a_1a_2+a_3a_4+a_1a_3+a_2a_4+a_1a_4+a_2a_3$ shall be squares.

*Solution by the* Proposer.

Let $$a_1 = x+y,\ \ a_2 = x-y,\ \ a_3 = y+z,\ \ a_4 = y-z;$$

then $$a_1a_2+a_3a_4 = x^2-y^2 = m^2, \text{ suppose};$$
$$a_1a_3+a_2a_4 = 2y\,(x+z) = n^2, \text{ suppose};$$
$$a_1a_4+a_2a_3 = 2y\,(x-z) = r^2, \text{ suppose};$$

therefore $$x = \tfrac{1}{4}m\,.\,(n^2+r^2)/nr,\ \ y = nr/2m,\ \ z = \tfrac{1}{4}m\,.\,(n^2-r^2)/nr;$$

Hence in integers we can take

$$x = m^2\,(n^2+r^2),\quad y = n^2r^2,\quad z = m^2\,(n^2-r^2);$$

therefore $$a_1 = m^2\,(n^2+r^2)+n^2r^2,\quad a_2 = m^2\,(n^2+r^2)-n^2r^2,$$
$$a_3 = n^2r^2+m^2\,(n^2-r^2),\quad a_4 = n^2r^2-m^2\,(n^2-r^2),$$

$$\therefore\quad a_1a_2+a_3a_4 = 4m^4n^2r^2,\quad a_1a_3+a_2a_4 = 4m^2n^4r^2,\quad a_1a_4+a_2a_3 = 4m^2n^2r^4.$$

Also, $$a_1a_2+a_3a_4+a_1a_3+a_2a_4+a_1a_4+a_2a_3 = 4m^2n^2r^2\,(m^2+n^2+r^2).$$

We have therefore to make $m^2+n^2+r^2$ a square.

Let $m^2+n^2+r^2 = (m+k)^2$; therefore $m = (n^2+r^2-k^2)/2k$.

Take $$n = 5s+9,\quad r = 5s+8,\quad k = 5.$$

Then $$m = 5s^2+17s+12 = (5s+12)(s+1).$$

Let $s = 0$, then $m = 12$, $n = 9$, $r = 8$;

therefore $$x = 144\times 145,\ y = 144\times 36,\ z = 144\times 17;$$

or, omitting the common factor 144,

$$x = 145,\quad y = 36,\quad z = 17.$$

Hence, $$a_1 = 181,\ a_2 = 119,\ a_3 = 53,\ a_4 = 19;$$

$$\therefore\quad a_1a_2+a_3a_4 = 144^2,\quad a_1a_3+a_2a_4 = 108^2,\quad a_1a_4+a_2a_3 = 96^2,$$
$$a_1a_2+a_3a_4+a_1a_3+a_2a_4+a_1a_4+a_2a_3 = 144^2+108^2+96^2 = 204^2.$$

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**10146.** (Professor Beyens.)—Résoudre le système des équations

$$p^2\,(q-r) = a,\quad q^2\,(r-p) = b,\quad r^2\,(p-q) = c.$$

*Solution by* W. J. GREENSTREET, M.A.; *and* G. G. STORR, M.A.

(1) × (2) × (3) + [(1) + (2) + (3)] gives us $p^2q^2r^2 = -\dfrac{abc}{a+b+c} = \lambda^2$ (say),

therefore $\quad pqr = \pm\lambda$ and $aq + bp = p^2q(q-r) + q^2p(r-p)$

$$= pqr(q-p) = \pm\lambda(q-p);$$

$$\therefore \quad \frac{p}{\pm\lambda - a} = \frac{q}{\pm\lambda + b}; \quad \text{similarly } r = p\left(\frac{\pm\lambda - c}{\pm\lambda + a}\right);$$

$$\therefore \quad \frac{p}{\pm\lambda - a} = \frac{q}{\pm\lambda + b} = \frac{r(\pm\lambda + a)}{\pm\lambda - c} = \sqrt[3]{\frac{\pm\lambda(\pm\lambda + a)}{(\pm\lambda - a)(\pm\lambda + b)(\pm\lambda - c)}};$$

whence $p, q, r$.

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**10083.** (CH. HERMITE.)—Démontrer que, pour $x > 1$, on a la relation $\log_e x > 2(x-1)/(x+1)$.

*Solution by* Rev. J. L. KITCHIN, M.A.; J. J. BARNIVILLE; *and others.*

Put $\quad x = \dfrac{m+1}{m-1}, \quad$ then $\dfrac{1}{m} = \dfrac{x-1}{x+1}, \quad$ and $x > 1, \quad$ if $m > 1$,

$$\therefore \quad \log x = \log\left(\frac{1+m^{-1}}{1-m^{-1}}\right) = 2\left(\frac{1}{m} + \frac{1}{3m^3} + \frac{1}{5m^5} + \dots\right) > \frac{2}{m}.$$

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**8974.** (W. J. C. SHARP, M.A.)—Show

(1) $S.\alpha\beta\gamma\delta = S.\alpha\delta\gamma\beta = S.\beta\alpha\delta\gamma =$ &c.;

(2) $S.(\beta\gamma V\gamma\alpha\beta + \gamma\alpha V\alpha\beta\gamma + \alpha\beta V\beta\gamma\alpha) = -S.\alpha\beta\gamma(S\beta\gamma + S\gamma\alpha + S\alpha\beta)$;

(3) $\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 3S\alpha\beta\gamma$.

*Solution by* D. EDWARDES, B.A.

(1) $S.\alpha\beta\gamma\delta = S.\alpha V\beta\gamma\delta = S.\alpha V\delta\gamma\beta = S.\alpha\delta\gamma\beta$

$\quad = S.\beta\alpha\delta\gamma = S.\beta V\alpha\delta\gamma = S.\beta V\gamma\delta\alpha = S.\beta\gamma\delta\alpha$, &c.

(2) $\beta\gamma\,\gamma\alpha\beta = \beta\gamma\,S\gamma\alpha\beta + \beta\gamma V\gamma\alpha\beta$; operating with S,

$0 = S.\beta\gamma\,S\gamma\alpha\beta + S.\beta\gamma V\gamma\alpha\beta$; therefore $S.\beta\gamma V\gamma\alpha\beta = -S\gamma\alpha\beta S\beta\gamma$.

Writing down two similar equations, and compounding the results, we have $S.(\beta\gamma V\gamma\alpha\beta + \gamma\alpha V\alpha\beta\gamma + \alpha\beta V\beta\gamma\alpha) = -S.\alpha\beta\gamma(S\beta\gamma + S\gamma\alpha + S\alpha\beta)$,

since $\quad S.\alpha\beta\gamma = S.\beta\gamma\alpha = S.\gamma\alpha\beta$.

(3) $\quad S.\alpha\beta\gamma = S.\alpha V\beta\gamma = \alpha V\beta\gamma - V.\alpha V\beta\gamma$,

or $\quad S.\alpha\beta\gamma = \alpha V\beta\gamma - \gamma S\alpha\beta + \beta S\gamma\alpha$;

similarly $\quad S.\beta\gamma\alpha = \beta V\gamma\alpha - \alpha S\beta\gamma + \gamma S\alpha\beta$,

$\quad S.\gamma\alpha\beta = \gamma V\alpha\beta - \beta S\gamma\alpha + \alpha S\beta\gamma$;

whence, adding, we have $\quad 3S.\alpha\beta\gamma = \alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta$.

**9792.** (W. J. C. SHARP, M.A.)—Show that, if
$(p_0x^n - p_1x^{n-1} + p_2x^{n-2}\dots)^m = q_0x^{mn} - q_1x^{mn-1} + q_2x^{mn-2} - q_3x^{mn-3} + \&c.,$

$$q_r = \frac{1}{r!}\left(p_1\frac{d}{dp_0} + 2p_2\frac{d}{dp_1} + 3p_3\frac{d}{dp_2} + \&c.\right)^r . (p_0)^m,$$

and that, if $m = 2$, $q_r = p_r . p_0 + p_{r-1} . p_1 + p_{r-2} . p_2 \dots + p_0 . p_r$.
Also deduce the expansion of $\phi(fx)$ in powers of $x$, where $\phi(x)$ is an integral and rational function of $x$, and

$$f(x) = p_0x^n - p_1x^{n-1} + p_2x^{n-3} - p_3x^{n-3} + \&c.$$

*Solution by* Prof. SIRCOM; D. EDWARDES, M.A.; *and others.*

Let $\qquad \phi(p_0 - p_1y + p_2y^2 - \dots) = A_0 - A_1y + A_2y^2 - \dots$

Then, since $\qquad \dfrac{d\phi}{dy} = (-p_1 + 2p_2y - 3p_3y^2 + \dots)\,\phi',$

and $\qquad \dfrac{d\phi}{dp_0} = \phi',\ \dfrac{d\phi}{dp_1} = -y\phi',\ \dfrac{d\phi}{dp_2} = y^2\phi',$ &c.,

we have $\qquad \dfrac{d\phi}{dy} = -\left(p_1\dfrac{d}{dp_0} + 2p_2\dfrac{d}{dp_1} + 3p_3\dfrac{d}{dp_2} + \dots\right)\phi;$

whence $\qquad -\left(\dfrac{d\phi}{dy}\right)_0 = A_1 = \left(p_1\dfrac{d}{dp_0} + 2p_2\dfrac{d}{dp_1} + 3p_3\dfrac{d}{dp_2} + \dots\right)\phi(p_0),$

and $\qquad A_r = \dfrac{1}{r!}\left(p_1\dfrac{d}{dp_0} + 2p_2\dfrac{d}{dp_1} + 3p_3\dfrac{d}{dp_2} + \dots\right)^r\phi(p_0),$

and the particular cases follow immediately, writing $1/x$ for $y$.

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**9796, 9777.** (W. J. C. SHARP, M.A.)—Show that the transformation from rectangular to areal coordinates, or *vice versâ*, may be effected by substitution from the equations

$$(\lambda+\mu+\nu)\,x = \lambda x_1 + \mu x_2 + \nu x_3, \quad (\lambda+\mu+\nu)\,y = \lambda y_1 + \mu y_2 + \nu y_3,$$

where $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are the vertices of the triangle of reference. And similarly, that the rectangular and tetrahedral coordinates of a point in space of three dimensions are connected by the equations

$$(\lambda+\mu+\nu+\pi)\,x = \lambda x_1 + \mu x_2 + \nu x_3 + \pi x_4,$$
$$(\lambda+\mu+\nu+\pi)\,y = \lambda y_1 + \mu y_2 + \nu y_3 + \pi y_4,$$

and $\qquad (\lambda+\mu+\nu+\pi)\,z = \lambda z_1 + \mu z_2 + \nu z_3 + \pi z_4;$

or more generally that, in space of $n$ dimensions, the connection between orthogonal and simplicissimum content coordinates (see Question 8242) is given by the equations

$$(\lambda+\mu+\nu\dots+\tau)\,x = \lambda x_1 + \mu x_2 + \dots + \tau x_{n+1},$$
$$(\lambda+\mu+\nu\dots+\tau)\,y = \lambda y_1 + \mu y_2 + \dots + \tau y_{n+1},$$
$$\&c. \qquad\qquad \&c.$$

9777. If $l\lambda + m\mu + n\nu + \dots = 0$ be the equation to a linear locus in space of $n$ dimensions, in terms of the simplicissimum content coordinates (areal, tetrahedral, &c.), (see Question 8242); show that $l, m, n$, &c. are

proportional to the perpendiculars drawn from the vertices of the simplicissimum of reference upon the locus.

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*Solution by* Professor SEBASTIAN SIRCOM.

9796. The determinant $\begin{vmatrix} x_1, & x_2 & \dots & x_{n+1}, & x \\ x_1, & x_2 & \dots & x_{n+1}, & x \\ y_1, & y_2 & \dots & y_{n+1}, & y \\ z_1, & z_2 & \dots & z_{n+1}, & z \\ \vdots & \vdots & & \vdots & \vdots \\ 1, & 1 & \dots & 1, & 1 \end{vmatrix}$

vanishes, but its minors are proportional to the contents of the simplicissima, the coordinates of whose vertices are involved, they are therefore proportional to $\lambda, \mu, \nu \dots$.

Similarly,

$\begin{vmatrix} 1, & 1, & 1, & 1, & 1 \\ x_1, & x_2, & \dots & x_{n+1}, & x \\ y_1, & y_2, & \dots & y_{n+1}, & y \\ z_1, & z_2, & \dots & z_{n+1}, & z \\ \vdots & \vdots & & \vdots & \vdots \\ 1, & 1, & \dots & 1, & 1 \end{vmatrix}$ vanishes; whence $\lambda+\mu+\nu+\dots$ is equal to $\begin{vmatrix} x_1, & x_2, & \dots & x_{n+1} \\ y_1, & y_2, & \dots & y_{n+1} \\ z_1, & z_2, & \dots & z_{n+1} \\ \vdots & \vdots & & \vdots \\ 1, & 1, & \dots & 1 \end{vmatrix}$,

or is proportional to the content of the fundamental simplicissimum: hence the result.

9777. If the equation of the linear locus in rectangular coordinates, $Ax+By+\dots+L=0$, be transformed to content coordinates by the substitution given above, we have

$$(Ax_1+By_1+Cz_1+\dots+L)\,\lambda+(Ax_2+By_2+Cz_2+\dots+L)\,\mu+\dots=0;$$

but $Ax_1+By_1+Cz_1+\dots+L$ is proportional to the perpendicular from $x_1, y_1, z_1 \dots$ on the linear locus, &c.

[These equations furnish a means of determining the sections of a locus, in space of $n$ dimensions, by any linear locus. They were used for that purpose by the proposer in a paper on the Sections of Surfaces, in a note on Surfaces represented by Quaternion Equations (Vol. XLIII., p. 47), and in the Solution of Quest. 7676 (Vol. XII., p. 105).]

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**10241.** (H. L. ORCHARD, M.A., B.Sc.)—Find the value of $k$, in order that $$4x^4 \dagger 4x^3 \dagger 5kx^2 \dagger 2x \dagger 1 = 0$$ may be soluble as a simple quadratic.

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*Solution by* R. KNOWLES, B.A.; EMILY PERRIN; *and others.*

The expression (see YOUNG'S *Algebra*) may be at once broken into the quadratic factors $\quad 2x^2+x+1 \pm (1-5k+4)\ x$;

hence, when $k=1$, the equation $2x^2+x+1=0$ will contain all the roots.

**10085.** (D. Edwardes, B.A.)—Given that
$x(a^2-m\beta\gamma)=y(\beta^2-m\gamma a)=z(\gamma^2-ma\beta)\equiv\lambda,\;\; m^2(a^3+\beta^3+\gamma^3)=(1+2m^3)a\beta\gamma,$
prove that $$a(x^2-myz) = \beta(y^2-mzx) = \gamma(z^2-mxy),$$
$$m^2(x^3+y^3+z^3) = (1+2m^3)\,xyz.$$

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*Solution by* Rev. J. L. Kitchin, M.A.; Professor Mukhopadhyay, M.A.; *and others.*

$$x = \lambda/(a^2-m\beta\gamma),\quad y = \lambda/(\beta^2-m\gamma a),\quad z = \lambda/(\gamma^2-ma\beta),$$

therefore $$x^2-myz = \lambda^2\left\{\frac{1}{(a^2-m\beta\gamma)^2}-\frac{m}{(\beta^2-m\gamma a)(\gamma^2-ma\beta)}\right\}$$
$$= \frac{\lambda^2}{\mathrm{P}(a^2-m\beta\gamma)}\left\{(\beta^2-m\gamma a)(\gamma^2-ma\beta)-m(a^2-m\beta\gamma)^2\right\},$$

where $$\mathrm{P} = (a^2-m\beta\gamma)(\beta^2-m\gamma a)(\gamma^2-ma\beta)$$
$$= \frac{\lambda^2}{\mathrm{P}}\,\frac{(m^3-1)}{m}\,\beta\gamma, \text{ on using the given condition.}$$

Therefore
$$a(x^2-myz) = \frac{\lambda^2}{\mathrm{P}}\,\frac{m^3-1}{m}\,a\beta\gamma = \beta(y^2-mzx) = \gamma(z^2-mxy) \text{ by symmetry.}$$

Hence $a, \beta, \gamma$; $x, y, z$ are interchangeable; therefore
$$m^2(x^3+y^3+z^3) = (1+2m^3)\,xyz.$$

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**10130.** (W. J. Greenstreet, M.A.)—Sum to $n$ terms the series
$$1^2\cos x+2^2\cos 2x+3^2\cos 3x+\ldots$$

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*Solution by* G. E. Crawford, B.A.; A. T. Warren; *and others.*

Let $\mathrm{C} = 1^2\cos x+2^2\cos 2x+\ldots n^2\cos nx$, and $\mathrm{S}$ = the like in sines, therefore $\mathrm{C}+i\mathrm{S} = 1^2e^{ix}+2^2e^{2ix}+\ldots n^2e^{nix} = 1^2y+2^2y^2+\ldots n^2y^n$, if $y=e^{ix}$. This is the recurring series whose scale of relation is $(1-y)^3$, and on multiplying we find
$$(1-y)^3(\mathrm{C}+i\mathrm{S}) = y+y^2+y^{n+1}\times\left\{-3n^2+3(n-1)^2-(n-2)^2\right\}$$
$$+y^{n+2}\times\left\{3n^2-(n-1)^2\right\}-y^{n+3}\,.\,n^2.$$

therefore $$\mathrm{C}+i\mathrm{S} = \frac{y+y^2-(n+1)^2y^{n+1}+(2n^2+2n-1)\,y^{n+2}-n^2y^{n+3}}{(1-y)^3}.$$

Now $$(1-y)^3 = (1-e^{ix})^3 = \left\{1-\cos x-i\sin x\right\}^3$$
$$= \left\{2\sin^2\tfrac{1}{2}x-2i\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x\right\}^3 = 8i\sin^3\tfrac{1}{2}x\,e^{\frac{3}{2}ix};$$

therefore C = imaginary part in
$$\frac{1}{8\sin^3\frac{1}{2}x\,e^{\frac{3}{2}ix}}\left\{e^{ix}+e^{2ix}-(n+1)^2e^{(n+1)ix}+(2n^2+2n-1)\,e^{(n+2)ix}-n^2e^{(n+3)ix}\right\}$$
$$= \frac{1}{4\sin^3\frac{1}{2}x}\left\{2n^2\sin(n+\tfrac{1}{2})\,x\sin^2\tfrac{1}{2}x+2n\sin\tfrac{1}{2}x\cos nx-\cos\tfrac{1}{2}x\sin nx\right\},$$

after a little work, replacing the exponentials by their trigonometrical values. Putting $n = 1$, this reduces correctly to $\cos x$; and putting $x = 0$ it reduces to $\frac{1}{6}n(n+1)(2n+1)$.

$$\left[\Sigma_{r=1}^{r=n} \cos rx = \frac{\cos\frac{1}{2}(n+1)x \sin\frac{1}{2}nx}{\sin\frac{1}{2}x}. \text{ Differentiate twice.}\right.$$

$$\left.\text{Therefore} \quad \Sigma_{r=1}^{r=n} r^2 \cos rx = -\frac{d^2}{dx^2}\left(\frac{\cos\frac{1}{2}(n+1)x\sin\frac{1}{2}nx}{\sin\frac{1}{2}x}\right) = \&c.\right]$$

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**10107 & 6313.** (Professor HUDSON, M.A.) — Prove, if R be the circum-radius, that the distance between the in-centre and the orthocentre of a triangle is

$$2R\left\{\text{vers A vers B vers C} - \cos A \cos B \cos C)\right\}^{\frac{1}{2}}.$$

*Solution by* Rev. D. THOMAS, M.A.; R. KNOWLES, B.A.; *and others.*

Vectors of in- and orthocentres (P) are

$$(a\alpha + b\beta + c\gamma)/\Sigma a, \quad \cot B \cot C\alpha + \cot C \cot A\beta + \cot A \cot B\gamma,$$

and vector $PI = (a/\Sigma a - \cot B \cot C)\alpha + \ldots$; $\alpha$, $\beta$, $\gamma$ being vectors of A, B, C originating at circumcentre.

Then $\quad PI^2 = \left[a/\Sigma a + b/\Sigma a + c/\Sigma a - \Sigma \cot B \cot C\right]^2$

$$- a^2(b/\Sigma a - \cot C \cot A)(c/\Sigma a - \cot A \cot B)$$

$$= -abc/\Sigma a + bc(b+c)/\Sigma a \cot B \cot C + \ldots - \cot A \cot B \cot C(a^2 \cot A + \ldots)$$

$$= -2abc/\Sigma a + bc \cot B \cot C - \cot A \cot B \cot C(a^2 \cot A + b^2 \cot B + c^2 \cot C)$$

$$= 4R^2(1 - \Sigma \cos A) + 4R^2 \Sigma \cos B \cos C - 8R^2 \cos A \cos B \cos C = \text{result}.$$

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**9523.** (ASPARAGUS.)—In the ambiguous case of the Solution of Triangles, the given angle is 60°; prove that the distance between the circumcentre and orthocentre of either of the two triangles is equal to the third side of the other triangle.

*Solution by the* PROPOSER.

Let $a$, $b$ be the given sides, $(a > b)$, and $B = 60°$, then, if $c$ be the third side, $b^2 = a^2 + c^2 - ac$, and the square of the distance between circumcentre and orthocentre is $\quad R^2(1 - 8\cos A \cos B \cos C)$,

or $\quad R^2(1 - 4\cos A \cos C) = R^2[1 - 2\cos(A-C) - 2\cos(A+C)]$

$$= 4R^2 \sin^2\tfrac{1}{2}(A-C),$$

and $\quad a - c = 2R(\sin A - \sin C) = 4R \sin\frac{1}{2}(A-C)\cos\frac{1}{2}(A+C)$

$$= 2R \sin\tfrac{1}{2}(A-C).$$

But the sum of the two values of $c$ is $a$, hence $a - c$ in one triangle is equal to the third side of the other triangle.

[The question may be otherwise given in the following form :—
The angle A of a triangle ABC is 60°; prove that the distance from the circumcentre to the orthocentre is equal to the difference of the sides AB, AC; and, when the angle A is 120°, the distance from the circumcentre to the orthocentre is equal to the sum of the sides AB, AC.

The distance from circumcentre to orthocentre is equal to $b \sim c$, if $2\cos A$ has either of the values $1$, $4\cos(B-C)-3$; and to $b+c$, if $2\cos A$ has either of the values $-1$, $4\cos(B-C)+3$.

The PROPOSER states that the theorem was discovered in calculating for the triangle in which $a = 775{\cdot}2704$, $b = 674{\cdot}2277$, $B = 60°$, in which $\Delta_1 = c_2 = 325{\cdot}9911$, $\Delta_2 = c_1 = 449{\cdot}2793$: an odd way of finding out so simple a theorem.]

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**9771.** (ASPARAGUS.)—In a triangle, the distance between the circumcentre and the orthocentre is equal to the difference of two of the sides $(a \sim b)$; prove that the angle $C = 60°$.

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*Solution by* W. J. GREENSTREET, M.A.; SARAH MARKS, B.Sc.; *and others.*

If $C = 60°$, $\Delta = \frac{1}{4}\sqrt{3}\,ab$; therefore $R^2 = \frac{1}{3}c^2$;

and $$1-4\cos A\cos B = 1-4\,.\frac{2b-a}{2c}\,.\,\frac{2a-b}{2c} = \frac{3(a-b)^2}{c^2};$$

therefore $$R^2(1-8\cos A\cos B\cos C) = \frac{c^2}{3}\,.\,\frac{3(a-b)^2}{c^2} = (a-b)^2.$$

therefore $$OP = a \sim b.$$

[This assumes the triangle to be finite and real; the stated property leads to the equation $(2\cos C-1)\left\{\cos^2\frac{1}{2}C+\sin^2\frac{1}{2}(A-B)\right\} = 0$.

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**10111.** (Professor WOLSTENHOLME, M.A., Sc.D.)—If $a, b, c, A, B, C$ denote the sides and angles respectively of a triangle, and $n$ be any positive integer; prove that

$$b^n\cos nC + nb^{n-1}c\cos[(n-1)C-B] + \tfrac{1}{2}(n\,.\,n-1)\,b^{n-2}c^2\cos[(n-2)C-2B$$
$$+\ldots+n\,bc^{n-1}\cos[C-(n-1)B] + c^n\cos nB = a^n,$$
$$b^n\sin nC + nb^{n-1}c\sin[(n-1)C-B] + \tfrac{1}{2}(n\,.\,n-1)\,b^{n-2}c^2\sin[(n-2)C-2B]$$
$$+\ldots+n\,bc^{n-1}\sin[C-(n-1)B] + c^n\sin(-nB) = 0.$$

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*Solution by* G. E. CRAWFORD, B.A.; A. T. WARREN; *and others.*

Taking C and S to represent the two series,

$$C+iS = (be^{iC}+ce^{-iB})^n = \left\{b(\cos C+i\sin C)+c(\cos B-i\sin B)\right\}^n$$
$$= (b\cos C+c\cos B)^n = a^n;$$

hence, equating real and imaginary parts, $C = a^n$, $S = 0$.

**10216.** (R. W. D. CHRISTIE.)—If $x$, $y$, $z$ be sides, and $a$, $b$, $c$ medians of a triangle, prove that

$$3^3(x^2+y^2+z^2) \equiv \{2(2a+2b-c)\}^2 + \{2(2b+2c-a)\}^2 + \{2(2c+2a-b)\}^2 \equiv (6a)^2+(6b)^2+(6c)^2.$$

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*Solution by* J. WOODALL; R. KNOWLES, B.A.; *and others.*

$$2(x^2+y^2) = 4c^2+z^2, \quad 2(y^2+z^2) = 4a^2+x^2, \quad 2(z^2+x^2) = 4b^2+y^2,$$

therefore $\quad 3(x^2+y^2+z^2) \equiv 4(a^2+b^2+c^2)$;

therefore $\quad 3^3(x^2+y^2+z^2) \equiv (6)^2(a^2+b^2+c^2)$

$$\equiv 2^2\{(2b+2c-a)^2+(2c+2a-b)^2+(2a+2b-c)^2\}.$$

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**9694.** (ASPARAGUS.)—If an equilateral triangle be inscribed in the ellipse $x^2/a^2+y^2/b^2=1$, prove that (1) the locus of the centre of the triangle is the ellipse $(a^2+3b^2)^2x^2/a^2+(b^2+3a^2)^2y^2/b^2=(a^2-b^2)^2$; and (2) if O be this centre, P the point where the circumcircle again meets the given ellipse, OP will be normal to the ellipse

$$x^2/a^2(a^2+3b^2)^2+y^2/b^2(b^2+3a^2)^2 = (a^2+b^2)^2/(a^2-b^2)^4,$$

which is the reciprocal of the last ellipse with respect to

$$x^2/a^2+y^2/b^2 = (a^2+b^2)/(a^2-b^2).$$

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*Solution by* Rev. T. GALLIERS, M.A.; G. G. STORR, M.A.; *and others.*

1. Let $\alpha$, $\beta$, $\gamma$ be the eccentric angles of the corners of the triangle, and suppose $\alpha+\beta+\gamma=\Sigma$. The necessary and sufficient condition that the triangle should be equilateral is that its centroid should coincide with its circumcentre.

Now the coordinates of the circumcentre and centroid are

$$\frac{a^2-b^2}{4a}\{\cos\alpha+\cos\beta+\cos\gamma+\cos\Sigma\}, \quad -\frac{a^2-b^2}{4b}\{\sin\alpha+\sin\beta+\sin\gamma-\sin\Sigma\},$$

$$\tfrac{1}{3}a(\cos\alpha+\cos\beta+\cos\gamma), \quad \tfrac{1}{3}b(\sin\alpha+\sin\beta+\sin\gamma).$$

Hence, if $(x, y)$ be the centre of the triangle,

$$(a^2+3b^2)x = a(a^2-b^2)\cos\Sigma, \quad (b^2+3a^2)y = b(a^2-b^2)\sin\Sigma;$$

hence locus of centre is $(a^2+3b^2)^2x^2/a^2+(b^2+3a^2)^2y^2/b^2=(a^2-b^2)^2 \ldots$ (1).

2. If $\delta$ be the eccentric angle of P, we know (see SALMON'S *Conics*, 5th Edition, Art. 244, Ex. 1) that $\alpha+\beta+\gamma+\delta=0$,

or P is $(a\cos\Sigma, -b\sin\Sigma)$, and OP will be found to be

$$y = -a(a^2+3b^2)\tan\Sigma\,.\,x/\{b(b^2+3a^2)\}+(a^4-b^4)\sin\Sigma/\{b(b^2+3a^2)\} \ldots (2),$$

the general equation of the normal to the ellipse $x^2/A^2+y^2/B^2=1$ is

$$y = Mx-(A^2-B^2)M/(B^2M^2+A^2)^{\frac{1}{2}} \;\ldots\ldots\ldots\; (3, 4),$$

It will be found that equation (2) will coincide with (4), when

$$M = -a\,(a^2+3b^2)\tan\Sigma\,/\,\{b\,(b^2+3a^2)\},$$

and $A = a\,(a^2+3b^2)\,(a^2+b^2)/(a^2-b^2)^2, \quad B = b\,(b^2+3a^2)\,(a^2+b^2)/(a^2-b^2)^2.$

Hence OP is normal to the ellipse

$$x^2/a^2\,(a^2+3b^2)^2 + y^2/b^2\,(b^2+3a^2)^2 = (a^2+b^2)^2/(a^2-b^2)^4 \;\ldots\ldots\ldots\; (5).$$

Lastly, it may be shown that the ellipse (5) is the reciprocal of the ellipse (1) with respect to the ellipses $x^2/a^2 + y^2/b^2 = (a^2+b^2)/(a^2-b^2)$. For the reciprocal indicated will be the envelope of the straight line

$$hx/a^2 + ky/b^2 = (a^2+b^2)/(a^2-b^2),$$

where $h$, $k$ are connected by the equation

$$(a^2+3b^2)^2\,h^2/a^2 + (b^2+3a^2)^2\,k^2/b^2 = (a^2-b^2)^2,$$

and the equation of this envelope will be found to be (5).

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**9558.** (H. Fortey, M.A.)—Rationalise (1) $x^{\frac{1}{7}}+y^{\frac{1}{7}}+z^{\frac{1}{7}}=0$, (2) $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}+u^{\frac{1}{3}}=0$, (3) $x^{\frac{1}{5}}+y^{\frac{1}{5}}+z^{\frac{1}{5}}+u^{\frac{1}{5}}+v^{\frac{1}{5}}=0$.

*Solution by the* Proposer.

1. Let $x^{\frac{1}{7}}$, $y^{\frac{1}{7}}$, $z^{\frac{1}{7}}$ be the roots of the cubic equation $t^3-at+b=0$; then (see Todhunter's *Theory of Equations*, 5th Edition, p. 189)

$$x+y+z=-7a^2b, \quad x^2+y^2+z^2=2a^7+35a^4b^2+14ab^4,$$

therefore $\qquad xy+yz+zx=-a^7+7a^4b^2-7ab^4;$

also $\qquad xyz=-b^7,$

therefore $\qquad \sigma_1=-7a^2b, \quad \sigma_2=-a^7+7a^4b^2-7ab^4, \quad \sigma_3=-b^7;$

and, eliminating $a$ and $b$ from these three equations, we get

$$\sigma_1^7 = 7^4\sigma_3\,(5\sigma_1^4-2\,.\,7^2\sigma_1^2\sigma_2-7^4\sigma_1\sigma_3+7^3\sigma_2^2),$$

the rational form.

2. Let the equation whose roots are $x^{\frac{1}{3}}$, $y^{\frac{1}{3}}$, &c., be $t^4+bt^2+ct+d=0$; then the equation whose roots are $x$, $y$, &c., is

$$t^{\frac{4}{3}}+bt^{\frac{2}{3}}+ct^{\frac{1}{3}}+d=0, \quad \text{or} \quad d+(t+c)\,t^{\frac{1}{3}}+bt^{\frac{2}{3}}=0;$$

and multiplying this twice in succession by $t^{\frac{1}{3}}$, and then eliminating linearly the radicals $t^{\frac{1}{3}}$, $t^{\frac{2}{3}}$, we have

$$\begin{vmatrix} d, & t+c, & b \\ bt, & d, & t+c \\ t\,(t+c), & bt, & d \end{vmatrix} = 0,$$

or $\qquad t + 3ct^3+(b^3-3bd+3c^2)\,t^2+(c^3-3bcd)\,t+d^3=0;$

hence, if $\Sigma x=\sigma_1$, $\Sigma xy=\sigma_2$, &c., we have

$$-\sigma_1=3c, \quad \sigma_2=b^3-3bd+3c^2, \quad -\sigma_3=c^3-3bcd, \quad \sigma_4=d^3.$$

Eliminating $b$, $c$, $d$ from these four equations, we get

$$(\sigma_1^3-27\sigma_3)^3 = -3.9^3\sigma_1^2\sigma_4(2\sigma_1^3-9\sigma_1\sigma_2+27\sigma_3),$$

the rational result.

3. Let $x^{\frac{1}{2}}$, $y^{\frac{1}{2}}$, &c. be the roots of $t^5+bt^3+ct^2+dt+e = 0$, then $x$, $y$, &c., will be the roots of $t^{\frac{5}{2}}+bt^{\frac{3}{2}}+ct+dt^{\frac{1}{2}}+e = 0$,

or $$(t^2+bt+d)\,t^{\frac{1}{2}}+ct+e = 0,$$

or $$t^5+2bt^4+(2d+b^2)\,t^3+(2bd-c^2)\,t^2+(d^2-2ce)\,t-e^2 = 0,$$

Therefore $-\sigma_1 = 2b$, $\sigma_2 = 2d+b^2$, $-\sigma_3 = 2bd-c^2$, $\sigma_4 = d^2-2ce$, $\sigma_5 = e^2$, and, eliminating $b$, $c$, $d$, $e$ from these five equations, we have the rational result,

$$\left\{(\sigma_1^2-4\sigma_2)^2-64\sigma_4\right\}^2 = 2.4^5\sigma_5(\sigma_1^3-4\sigma_1\sigma_2+8\sigma_3).$$

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**10179.** (John J. Barniville.)—Prove that (1) the continued surd

$$\sqrt{a+b\sqrt{a+b\sqrt{a+\&c.}}} = \frac{a}{-b+}\;\frac{a}{-b+}\;\frac{a}{-b+},\ \&c.$$

(2) if $\sqrt{a+b\sqrt{a+b\sqrt{\&c.}}} = X$, and $\sqrt{a-b\sqrt{a-b\sqrt{a-\&c.}}} = Y$,

then $XY = a$, and $X-Y = b$; (3) $2\cos\frac{1}{5}\pi = \sqrt{1+\sqrt{1+\sqrt{1+\&c.}}}$,

and $$2\sin\tfrac{1}{10}\pi = \sqrt{1-\sqrt{1-\sqrt{1-\&c.}}} = \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+\&c.};$$

$$(4)\quad i = \sqrt{i-1-\sqrt{i-1-\&c.}},$$

and $$\omega \text{ or } \omega^2 = \sqrt{-1-\sqrt{-1-\sqrt{-1-\&c.}}} = \frac{-1}{1+}\;\frac{-1}{1+}\;\frac{-1}{1+\&c.}.$$

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*Solution by* R. H. W. Whapham, B.A.

(1) Let $$x = \sqrt{a+b\sqrt{a+b\sqrt{a+\dots}}} = \sqrt{a+bx};$$

$$\therefore\ x^2-bx-a = 0.$$

Let $$y = \frac{a}{-b+}\;\frac{a}{-b+}\;\frac{a}{-b+}\dots = \frac{a}{-b+y},$$

$$\therefore\ y^2-by-a = 0,\quad \therefore\ x = y.$$

(2) We have $$X^2-bX-a = 0,\quad Y^2+bY-a = 0;$$

$$\therefore\ (X+Y)(X-Y-b) = 0,\quad \therefore\ X-Y = b,$$

also $$X^2Y-bXY-aY = 0;\quad XY^2+bXY-a = 0;$$

$$\therefore\ XY(X+Y)-a(X+Y) = 0,\quad \therefore\ XY = a.$$

(3) Let $$x = \sqrt{1+\sqrt{1+\sqrt{1+\dots}}} = \sqrt{1+x};$$

$\therefore\ x^2-x-1 = 0$; $\therefore\ x =$ positive root $= (1+\sqrt{5})/2 = 2\cos\pi/5.$

Let $$y = \sqrt{1-\sqrt{1-\sqrt{1-\dots}}} = \sqrt{1-y};$$

$\therefore\ y^2+y-1=0;\ \ \therefore\ y = \text{positive root} = (\sqrt{5}-1)/2 = 2\sin\pi/10.$

Also $$y = \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+\dots}\ \text{by (1)}.$$

(4) Let $$x = \sqrt{i-1-\sqrt{i-1-\dots}} = \sqrt{i-1-x};$$
$$x^2+x = i-1;\ \ \therefore\ x = -1+(1+2i) = i.$$

Let $$y = \sqrt{-1-\sqrt{-1-\dots}} = \sqrt{-1-y};$$
$$\therefore\ y^2+y+1=0,\ \ \therefore\ y = \omega \text{ or } \omega^2.$$

Also $$y = \frac{-1}{1+}\ \frac{-1}{1+}\ \frac{-1}{1+\dots}\ \text{by (1)}.$$

[An expression may be turned into a continued surd by the formula
$$x = \sqrt[n]{x^n-px+p\sqrt[n]{x^n+px-\&c.}} = \sqrt[n]{x^n+px-\sqrt[n]{x^n+px-\&c.}}$$
Thus $$1 = \sqrt[n]{2-\sqrt[n]{2-\sqrt[n]{2-\&c.}}} = \sqrt[n]{0+\sqrt[n]{0+\sqrt[n]{0+\&c.}}}$$
$$= -\sqrt[2n]{0-\sqrt[2n]{0-\&c.}} = \frac{0}{-1+}\ \frac{0}{-1+}\ \frac{0}{-1+\&c.};$$
$$2 = \sqrt{2+\sqrt{2+\sqrt{2+\&c.}}} = \sqrt{6-\sqrt{6-\sqrt{6-\&c.}}}$$
$$= \sqrt[3]{6+\sqrt[3]{6+\sqrt{6+\&c.}}} = \sqrt[6]{66-\sqrt[6]{66-\sqrt[6]{66-\&c.}}} = \&c.]$$

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**9979.** (Professor Wolstenholme, M.A., Sc.D. Suggested by Quest. 9587, Vol. 50, p. 117).—In a triangle ABC, CC′ is the median through C, CS a chord of the circumcircle along the symmedian through C; the parabola whose focus is S and directrix CC′ will touch the side BC, the straight lines through A, B, at right angles to CA, CB, and the two bisectors of the angle C and its supplement. [The trilinear equation is
$$2(-\gamma)^{\frac{1}{2}} + [(\alpha+\beta)(\cos B+\cos A)]^{\frac{1}{2}} + [(\alpha-\beta)(\cos B-\cos A)]^{\frac{1}{2}} = 0.]$$

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*Solution by the* Proposer.

Since the equation of the cubic $(\alpha^2-\beta^2)+2\alpha\beta\,(\alpha\cos A-\beta\cos B)=0$ may be written
$$(\alpha\sin A+\beta\sin B+\gamma\sin C)(\alpha^2-\beta^2) = (\alpha^2+\beta^2+2\alpha\beta\cos C)(\alpha\sin A-\beta\sin B),$$
it passes through the isotropic points, or is a *circular* cubic; and its real asymptote is parallel to $\alpha\sin A = \beta\sin B$, which is the median through C of the triangle ABC. But every nodal circular cubic is a pedal of a parabola, the node being the origin of the pedal, and the asymptote is parallel to the directrix. In this case, the nodal tangents are at right angles, so that the directrix of the parabola passes through the node, and must, therefore, be the median through C. This parabola touches the bisectors of the angle C, also the side AB, and the straight lines drawn

through A, B at right angles to CA, CB respectively (because the parabola is the first negative pedal of the cubic with respect to C). Hence, the focus of the parabola is the intersection of the circle ABC and the circle circumscribing the triangle formed by AB and the two bisectors of C. This construction may be easily proved to be equivalent to:—the symmedian line through C meets the circumcircle of ABC in the form of the parabola. Thus the parabola is completely determined, the form being $\alpha/\sin A = \beta/\sin\beta = -2\gamma/\sin C$, and the directrix $\alpha \sin A = \beta \sin B$. The cubic is the pedal of this parabola with respect to C.

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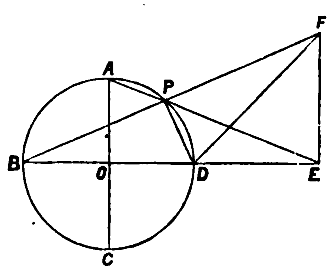
**10333.** (W. J. Greenstreet, M.A.)—AC, BD are fixed diameters at right angles to each other, and P any point on the circumference of the circle ABCD. PA cuts BD in E; EF parallel to AC cuts PB in F; prove that the locus of F is a straight line.

*Solution by* C. Morgan, M.A.; Rev. W. J. Constable, M.A.; *and others.*




A circle will go round EDPF, hence we have

$\angle EDF = EPF = APB = \frac{1}{4}\pi$;

therefore the locus of F is the straight line DF, which makes an angle of $\frac{1}{4}\pi$ with BD.

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**10259.** (Professor de Longchamps.)—On considère une conique H et un point fixe M. De M, comme centre, avec un rayon variable, on décrit un cercle $\Gamma$. Démontrer que le lieu des points de rencontre des tangentes communes à H, $\Gamma$ est une Strophoïde oblique.

*Solution by* Professor Schoute.

When two common tangents of H and $\Gamma$ concur in P, the join MP of P and the centre M of $\Gamma$ is in P normal to the one and tangent to the other of the two conics through P confocal to H. So the locus in question can also be considered as the locus of the feet of the normals through M to the conics confocal with H, or as the locus of the points of contact of the tangents through M to these conics. [For a geometrical demonstration, that this locus is a skew strophoïd, see Crelle's *Journal*, Vol. xcix., p. 106.]

# APPENDIX.

## UNSOLVED QUESTIONS.

3494. (The late T. Cotterill, M.A.) — 1. If $\rho = 0$, $\sigma = 0$ be respectively the equations to conics through A and B, and intersecting again in four points; $X = 0$, $Y = 0$ the equations to two cubics, the first nodal at A and passing through the other five points, the second nodal at B and passing through the other five points; then $\rho Y + a\sigma X = 0$ is the equation to a quintic curve nodal at the six points and passing through C the remaining intersection of the two cubics.

2. Find a line through C to any point of which a point on the quintic corresponds, so that an infinite number of cubics can pass through the pairs of points and the seven points above mentioned.

3. The envelope of the line joining corresponding points is the point C and a conic touching the line.

4077. (The late T. Cotterill, M.A.)—If $\alpha$, $\beta$, $\gamma$ be the sines of the great arcs drawn from a point on a sphere perpendicular to the sides of the spherical triangle ABC, prove (1) that

$$\tan a \cos A \,.\, \alpha^2 + \tan b \cos B \,.\, \beta^2 + \tan c \cos C \,.\, \gamma^2 = 0$$

is the equation to the small circles to which ABC is a self-conjugate triangle; (2) that its centre H is the intersection of the great circles through the angles perpendicular to the opposite sides; and (3) that if AH meet BC in D, BH in E, and CH in F, its radius ($\rho$) is given by the equations

$$\tan^2 G = \tan HA \tan HD = \tan HB \tan HC \,.\, \cos BHC$$
$$= \tan HE \,.\, \tan HF \sec BHC.$$

Also show (4) that the circle is real, if a side and opposite angle have contrary affections, and explain its relations to the supplemental triangle, and the cases when it degenerates or becomes indeterminate.

4079. (Professor Minchin, M.A.)—If $\alpha$, $\beta$, $\gamma$ be three coplanar or non-coplanar vectors, give a geometrical construction applicable in each case for $V\alpha\beta\gamma$.

4083. (Professor Hudson, M.A.)—A uniform wedge, whose section perpendicular to its edge is everywhere an isosceles triangle of which the semi-vertical angle is $\tan^{-1} 2^{\frac{1}{2}}$ and base $b$, floats with its edge fixed in the surface of a fluid of twice its specific gravity; it is then depressed through a small angle $\beta$ about the vertex: prove that the time in which it will return to its original position is approximately

$$\frac{1}{8\beta}\left(\frac{5b}{\pi g}\right)^{\frac{1}{2}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2.$$

4084. (Professor EVANS, M.A.)—A marksman is observed to plant $n$ per cent. of his arrows within a circle one foot in diameter at the distance of one hundred yards; find how many per cent. of his arrows can he plant in the same target at the distance of fifty yards.

4085. (Sir R. STAWELL BALL, F.R.S.)—If the two biquadratics

$$(a, b, c, d, e)(x, 1)^4 = 0 \text{ and } (a', b', c', d', e')(x, 1)^4 = 0$$

have the same invariants, and if $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ be their roots, then

$$a(\alpha-\gamma)(\beta-\delta)+a(\alpha-\delta)(\beta-\gamma) = a'(\alpha'-\gamma')(\beta'-\delta')+a'(\alpha'-\delta')(\beta'-\gamma'),$$

$$a(\alpha-\delta)(\gamma-\beta)+a(\alpha-\beta)(\gamma-\delta) = a'(\alpha'-\delta')(\gamma'-\beta')+a'(\alpha'-\beta')(\gamma'-\delta'),$$

$$a(\alpha-\beta)(\delta-\gamma)+a(\alpha-\gamma)(\delta-\beta) = a'(\alpha'-\beta')(\delta'-\gamma')+a'(\alpha'-\gamma')(\delta'-\beta').$$

4086. (A. RENSHAW.)—If AC be the chord of a heptagon inscribed in a circle, diameter AB ($=2r$), and AB be taken as the axis of $x$ and a line at right angles to it through A as that of $y$; then, if $(x', y')$ be the coordinates of C, prove that

$$-\frac{2y'(2x'-r).\{x'(x'-r)-y'^2\}}{\{x'(x'-r)-y'^2\}^2+y'^2(2x'-r)^2} = \frac{x}{y'}.$$

4087. (W. S. B. WOOLHOUSE, F.R.A.S.)—Suppose there to be a promiscuous series of numbers in which the decimals have been cut off and adjusted to the nearest units of the terminal figures. Then on summing $n-1$ numbers, arbitrarily taken, if from the arithmetical mean of the probabilities of the accumulated error respectively exceeding $e-\frac{1}{2}$ and $e+\frac{1}{2}$, of those units, there be subtracted $\frac{e}{n}\times$ probability of the error falling between $e-\frac{1}{2}$ and $e+\frac{1}{2}$, the difference will be the probability that the error will exceed $e$ units when a summation includes $n$ values.

4090. (The Rev. A. F. TORRY, M.A.)—A plate of Iceland spar is bounded by planes perpendicular to the axis of the crystal; light is incident upon it nearly in the direction of the axis; find the difference of retardation of the ordinary and extraordinary waves.

4091. (The late M. COLLINS, LL.D.)—Prove that

$$\frac{16N+20N^2D+5D^2}{16N^4+12N^2D+D^2}N$$

is nearer to $(N^2+D)^{\frac{1}{2}}$ than any rational fraction in its lowest terms $\frac{n}{d}$ when $d<16N^4+12N^2D+D^2$ and $D=\pm 1$; and show more generally that this theorem is true, whatever be the value of D, if

$$dD < 16N^4+12N^2D+D^2.$$

4099. (D. TROWBRIDGE, M.A.)—Prove that

$$1+\frac{1}{1+r}+\frac{1}{1+2r}+\ldots+\frac{1}{1+(n-1)r}$$

$$=n-\frac{n(n-1)r}{1.2(1+r)}+\frac{n(n-1)(n-2)2r^2}{1.2.3(1+r)(1+2r)}-\frac{n(n-1)(n-2)(n-3)3.2r^3}{1.2.3.4(1+r)(1+2r)(1+3r)}+\ldots$$

$$\ldots+(-1)^{\rho}\frac{n(n-1)(n-2)\ldots(n-\rho).\rho(\rho-1)(\rho-2)\ldots3.2r^{\rho}}{1.2.3\ldots(\rho+1).(1+r)(1+2r)\ldots(1+\rho r)}.$$

4100. (SAMUEL ROBERTS, M.A., F.R.S.)—Show that the number of normals which can be drawn from an arbitrary point to a surface which touches the plane at infinity at $p$ points, but is otherwise not specially related to that plane, is given by Order + Class + Rank $-p$.

4103. (The Rev. J. BLISSARD.) — If B is the representative of Bernoulli's numbers, prove that

$$(1) \ldots\ldots L^2 (1+\Delta)\, 0^n \left( = \frac{\Delta 0^n}{1^2} - \frac{\Delta^2 0^n}{2^2} + \frac{\Delta^3 0^n}{3^2} - \ldots \pm \frac{\Delta^n 0^n}{n^2} \right) = B_{n-1};$$

$$(2) \ldots\ldots \frac{\Delta 0^n}{2^2} - \frac{\Delta^2 0^n}{3^2} + \frac{\Delta^3 0^n}{4^2} - \ldots + \frac{\Delta^n 0^n}{(n+1)^2} \ (n \text{ odd}) = -\frac{n-2}{4} B_{n-1};$$

$$(3) \ldots\ldots L^3 (1+\Delta)\, 0^n \left( = \frac{\Delta 0^n}{1^3} - \frac{\Delta^2 0^n}{2^3} + \ldots - \frac{\Delta^n 0^n}{n^3} \right) (n \text{ even}) = \frac{n+1}{4} B_{n-2};$$

$$(4) \ldots\ldots \frac{\Delta 0^n}{3} - \frac{\Delta^2 0^n}{4} + \frac{\Delta^3 0^n}{5} - \ldots - \frac{\Delta^n 0^n}{n+2} \ (n \text{ even}) = -B_n.$$

4104. (Professor EVANS, M.A.)—A tetrahedron ABCD is cut by a plane that passes through A′, C′, the middle points of two opposite edges; prove (1) that A′C′ bisects the quadrilateral section A′B′C′D′; and (2) that the quadrilateral A′B′C′D′ divides the tetrahedron into two equivalent solids.

4105. (Professor HUDSON, M.A.) — A wheel in the form of a cylinder of radius R and thickness A has an axle of radius $r$ and length $a$ cut out of the same piece, the axes and centres of gravity being coincident. The whole is suspended with the axis horizontal by three vertical strings, one of which is coiled round the wheel and the other two round the axle at equal distances on either side of the wheel: prove that, if the first string be drawn up or let down in any way, the tensions of the other two will not be altered, provided $\dfrac{a}{A} = \dfrac{R^3 (R-2r)}{r^3 (2R-r)} + 1$.

4106. (M. GARDINER.) — When the solution of the problem,— To inscribe in a given surface of the second order a polygon whose sides shall pass in a given order of sequence through a given system of $n$ arbitrary points, — is determinate, the addition to the system of any point taken arbitrarily on the chord of determinate solution will render partially indeterminate the solution of the same problem for the $(n+1)$ points; and the further addition to the system of the pole with respect to the surface of the plane of partially indeterminate solution in the latter case will render wholly indeterminate the solution of the same again for the $(n+2)$ points. When all the points of the given system lie in the same plane, the solution for any order of sequence is never determinate; being partially or wholly indeterminate according as that of the same problem for the conic in which the plane of the system intersects the surface is determinate or indeterminate.

4108. (The late HENRY BUCKLEY.)—From P, one of three given points (P, Q, R), describe a circle, such that, if a tangent be drawn to it from Q to touch the circle in S, QS + RS may be the least possible.

4109. (ARTEMAS MARTIN, LL.D.)—A straight vertical tree on the side of a mountain whose elevation is $\beta$ is broken by the wind (which blows in an unknown direction), but not severely; find the chance that the top reaches the ground.

4110. (Dr. CHARLES TAYLOR.)—From two fixed points on a given conic pairs of tangents are drawn to a confocal conic, and with the fixed points as foci a conic is described passing through any one of the four points of intersection. Show that its tangent or normal at that point passes through a fixed point.

4111. (J. M. GREENWOOD, M.A.)—A plastic sphere 8000 miles in diameter, and of the same density as the earth, begins to revolve on its axis with a velocity such as to cause it to assume the shape of an oblate spheroid, whose polar diameter is to its equatorial diameter as $n$ is to $m$; find (1) the time of a revolution; and (2) find the time when $n=3$, $m=4$.

4112. (C. H. HINTON, M.A.)—Given two points, one on each side of a triangle, and the distances of each from the base-vertex of that side; given also the angle at which the line joining these points cuts the base and the ratio of the sides: construct the triangle.

4113. (W. SIVERLY.)—A hammer of given weight and dimensions, and supposed perfectly hard, falls from a height $h$ and impinges on a spring supposed perfectly elastic, and capable of being compressed $l$ feet, requiring $k$ pounds for each foot of compression. Required the maximum pressure on the spring.

4114. (A. RENSHAW.)—If OZ be a line drawn outside a circle whose centre is R, and RH the perpendicular from R on OZ; also, if $P_1Q$, $P_2R$, $P_3S$ be drawn at right angles to OZ from any three points $P_1$, $P_2$, $P_3$ on the circle, prove that

$$2HR\,(OS.P_1W-OQ.P_2Z-OT.P_1F)=OP_1^2.TS-OP_2^2.QT-OP_3^2.QS,$$

where $P_1W$, $P_2Z$, $P_1F$ are the differences of the perpendiculars from $P_1P_3$, $P_2P_3$, $P_1P_2$ and TS, QT, QS the segments of OZ between the feet of perpendiculars on OZ.

4123. (The late Professor CLIFFORD, F.R.S.) — The circles doubly normal to a bicircular quartic arrange themselves in four systems, each system cutting orthogonally a principal circle. It is required to find the envelope of all the binormal circles of one system.

4124. (Rev. A. F. TORRY, M.A.)—A certain transparent substance transmits only three of the colours of the spectrum, and these it absorbs partially and unequally. A ray of light traversing in succession a number of equal plates of this substance, it is found that the ray emerging from the first plate consists half of the first colour, and of equal parts of the two others, whilst that emerging from the second plate consists half of the second colour and of equal parts of the two others. In what proportions will the colours be present in the ray which emerges from the third plate?

4125. (Professor HUDSON, M.A.)—If a line join the points of contact of an escribed circle with the produced sides of a triangle, and corresponding lines be drawn for the other escribed circles so as to form an outer triangle; prove that the lines joining corresponding vertices of the two triangles are perpendicular to the sides of the former, and that they are equal to the radii of the escribed circles. Also, if from the outer triangle another triangle be formed in the same way, and so on, prove that these triangles tend to become equilateral.

4131. (Sir R. STAWELL BALL, F.R.S.)—Express the six roots of the equation

$(3abc-a^2d-2b^3)\,x^6+(9ac^2-6b^2c-2abd-ea^2)\,x^5+(15acd-5abe-10b^2d)\,x^4$
$+(10ad^2-10b^2e)\,x^3+(10bd^2+5ade-15bce)\,x^2+(6cd^2+2bde+ae^2-9c^2e)\,x$
$+2d^3-3cde+e^2b=0$ in terms of the roots $\alpha,\ \beta,\ \gamma,\ \delta$ of the equation $ax^4+4bx^3+6cx^2+4dx+e=0$.

4132. (The late M. COLLINS, LL.D.)—The equation of a curve of the $n$th order being $P+Q+R+S+\&c.=0$, where $P=ax^{n-2}y^2+a'x^{n-3}y^3+\&c.$, $Q=bx^{n-1}+b'x^{n-2}y+\&c.$, $R=cx^{n-2}+c'x^{n-3}y+\&c.$; show that Euler has fallen into error in giving $ay^2+bx=0$ as the equation of the asymptotic parabola, and find the *correct* equation.

4134. (The late T. COTTERILL, M.A.)—If M and $t$ are the point and direction ordinates of the line $t$, the curve $M=r\sin nt$ ($n$ being a real number) has at least one real apse and cusp. Explain the difference of form of the cusp as $n>$ or $<1$. The radii of the apsidal and cuspidal circles (centre M) are $r$ and $nr$. As a roulette, the fixed circle is always the same, viz., the apsidal circle for all values of $n$ can be generated as a hypocycloid, the radius of the movable circle being $\frac{1}{2}(n+1)\,r$. If $n>1$, it is also another hypocycloid, the radius of the movable circle being $\frac{1}{2}(n-1)\,r$. If $n<1$, it is an epicycloid, the radius of the movable circle being $\frac{1}{2}(1-n)\,r$. In the hypo- (or epi-) system, a point on the curve divides the chord of the apsidal circle cut off from the tangent ex- (or in-) ternally, and the chord of the cuspidal circle cut off from the normal in- (or ex-)ternally, in the ratio of the radii of the movable circles. Hence the normal cuts the curve and touches its evolute in harmonic conjugates to its intersections with the cuspidal circle. If the movable circles are drawn through the point, explain the similar relations between the arcs cut off from them and the apsidal circle by the tangent, and also those cut off from them and the cuspidal circle by the normal.

4136. (A. RENSHAW.)—On a billiard board three balls are placed in succession on the spot, and struck at random; find the chances that when at rest they shall (1) lie in the circumference of a circle that lies wholly within the table; (2) that the product of their distances from each other shall be less than the product of the sides of the table; (3) that the sum of the squares of their distances from the four sides (12 perpendiculars altogether) shall be less than the sum of the squares of the four sides of the table; (4) that the radius of the circle inscribed in the triangle they form shall be less than half one end of the table; (5) that they form an acute-angled triangle; also (6) suppose, on assuming a position of rest after the first stroke on each, they are again struck at random, what are the chances for the same things happening; and (7) supposing them to be struck at random $n$ times in succession after resting, what are the same chances for the same final results?

4141. (Professor CROFTON, F.R.S.)—Show that it is impossible to cut a given closed convex curve in more than two points by any circle whose radius is greater than the greatest, or less than the least radius of curvature of the curve.

4155. (HUGH MCCOLL, B.A.)—Let

$$f(x)=A_n x^n+A_{n-1}x^{n-1}+\ldots+A_1x+A_0,$$
$$f_1(x)=A_{n-1}x^{n-1}+2A_{n-2}x^{n-2}+3A_{n-3}x^{n-3}+\ldots+(n-1)\,A_1x+nA_0.$$

In Sturm's theorem substitute $f_1(x)$ for the first derived function $f'(x)$, and deduce the remaining auxiliary functions $f_2(x), f_3(x) \dots f_m(x)$ from $f(x)$ and $f_1(x)$ instead of from $f(x)$ and $f'(x)$. Let $a$ and $b$ be any two quantities *of the same sign*, of which $a$ is *numerically* greater than $b$. Show that if the series of functions $f(x), f_1(x), f_2(x) \dots f_m(x)$ give $p$ changes of signs when $x = a$, and $q$ changes when $x = b$, the number of real roots of $f(x) = 0$ between $a$ and $b$ is exactly $p - q$.

[In this theorem one change of sign is gained as $x$ increases through a positive root of $f(x) = 0$, or decreases algebraically through a negative root; while in Sturm's theorem one change of sign is lost as $x$ increases algebraically through any root. The same difference exists between Question 1739 and Fourier's theorem.]

4156. (D. Trowbridge, M.A.)—Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} = n - \frac{n(n-1)}{1 \cdot 2^2} S_2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3^2} S_3$$

$$- \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4^2} S_4 + \dots \pm \frac{n(n-1)(n-2)\dots 2 \cdot 1}{1 \cdot 2 \cdot 3 \dots n^2} S_n,$$

where $$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + n^{-1}.$$

4158. (J. F. Moulton, M.A.)—A lamina moves in its own plane so that two points fixed in the lamina describe straight lines with equal accelerations. Prove that the acceleration of the instantaneous centre is constant in direction, as is that of any point fixed in the lamina.

4159. (N'Importe.)—Show by which of the known methods the distance of the sun may be deduced from the transit of Venus with the greatest mathematical accuracy.

4160. (C. H. Hinton, M.A.)—If there be two circles cut by a straight line, and the points where each circle is cut be joined to its centre and the joining line produced,—(1) show that the four triangles thus formed on portions of the straight line as bases have their sides all in the same ratio; (2) how must the straight line move that the triangles thus formed have their sides in the same ratio for every position.

4161. (E. McCormick.)—If two cubics have a six-point contact at P, prove that they will have a six-point contact at another point Q that lies upon their common tangent at P.

4163. (The late M. Collins, LL.D.) — Show that the curves $U_1 + U_2 + U_3 = 0$ and $U_1 + U_2 + U_3 + U_1^2(Ax + By) = 0$ have the same osculating conic at the origin of coordinates, whatever be the values of the arbitrary constants A and B. Show also that the curves $U_1 + U_2 + U_3 + U_4 = 0$ and

$$U_1 + U_2 + U_3 + U_4 + U_1^2(Ax + By) + U_1(Cx^3 + Dx^2y + Exy^2 + Fy^3) = 0$$

have the same conic osculating both of them at the origin; whatever be the values of the six constants A, B, ... F. Extend and generalize these results.

4169 (Professor Sylvester, F.R.S.)—If in an equation of the fifth degree all the invariants, but not all the coefficients, are real, prove that real roots enter into such equation in pairs, and determine the invariantive conditions in order that an omni-reali-invariantive quintic may have 0, 2, or 4 real roots.

4171. (The late Professor CLIFFORD, F.R.S.)—The sides of a triangle repel with a force varying inversely as the cube of the distance; find the position in which a particle will rest. Also, supposing the faces of a tetrahedron to repel according to the same law, find where a particle will rest.

4177. (H. S. MONCK, M.A.)—The three sides of a right-angled triangle are commensurable; prove that neither of the angles at the base can be less than $\sin^{-1}\frac{3}{5}$ or $\tan^{-1}\frac{3}{4}$.

4178. (R. TUCKER, M.A.)—In the ambiguous case of plane triangles ($a$, B being fixed and $b$ variable), find the mean area of the triangle contained by the base and the median lines from C.

4181. (Professor CHRISTINE LADD, B.Sc.)—One end of a string, of length $a$ feet, is fastened to one end of a uniform rod, length $b$ feet and specific gravity one-$n$th that of water, and the other end of the string is fastened to a point $c$ feet above the surface of a deep river running with a velocity $v$, the lower end of the rod being in the water. Determine the rod's position of equilibrium.

4187. (W. SIVERLY.)—If $n$ bricks are piled at random on a horizontal plane, as they would be in a wall, find the chance that the pile will stand.

4189. (ARTEMAS MARTIN, LL.D.)—A uniform rod is suspended by a string attached to its middle point. A mouse runs down the string and along the rod to one end, with a uniform velocity. Find the equation to the curve the mouse describes in space.

4193. (Professor EVANS, M.A.)—At the station A, the apparent angular elevation of a meteor B, whose distance from the earth's surface is one-$n$th of the earth's radius, is $\theta$. Supposing the earth to be a perfect sphere, find the exact distance from A to B.

4194. (The EDITOR.)—Find the projective equation of the curve whose tangential equation is $a^2v^2 + b^2\xi^2 = a^2b^2(\xi^2 + v^2)^2$,

or the curve touched by one side of a right angle which moves along an ellipse, while the other side passes through the centre (BOOTH's *Tangential Coordinates*, p. 143).

4195. (Professor SYLVESTER, F.R.S.)—Given that the word ἑξείης occurs $m$ times in the Iliad and $\mu$ times in the Odyssey, but the contracted form ἑξῆς never in the Iliad and $\nu$ times in the Odyssey, discuss the proper mode of bringing these facts to bear upon the question of the identity of the authors of the two poems. Supposing $\nu$ to be equal to unity and $n$ to be equal to 289, ought it to be "convincing" (as, in the *Athenæum* for September 6, 1873, the author of the article on the doctrine of the Chorizontes in the *Edinburgh Review* alleges it would be to him) "that when the Iliad was written no such contraction was known in the Greek language"?

4202. (Sir R. STAWELL BALL, F.R.S.)—If in an equation $x$ be changed into $k + 1/x'$, show that any semi-invariant of the transformed will be a covariant in $k$ of the original equation.

4203. (J. C. MALET, M.A.)—The solutions of the linear differential equation $$\frac{d^3y}{dx^3}+\mathrm{P}\frac{d^2y}{dx^2}+\mathrm{Q}\frac{dy}{dx}+\mathrm{R}=0$$ being $y=y_1$, $y=y_2$, $y=y_3$, find the differential equation which has for a solution $y=\dfrac{ay_1+by_2+cy_3}{dy_1+ey_2+fy_3}$, where $a, b, c, d, e, f$ are arbitrary constants.

4211. (ELIZABETH BLACKWOOD, B.Sc.)—A point is taken at random in the surface of a circle, and a random line drawn through it; two other points are then taken at random in its surface: find the chance that they are on opposite side of the line.

4212. (W. SIVERLY.)—A cylinder, of length $l$ and diameter $a$, rests with its upper end against a tack in the side of a vessel containing water, in which the lower end floats. The greatest and least lengths of the immersed portion are $b+c$ and $b$. Find the weight of the cylinder.

4213. (Professor HUDSON, M.A.)—A very long row of particles, each of mass $m$, on a smooth horizontal table, are connected, each with the two adjacent ones, by similar light elastic stretched strings, each of natural length $c$; they receive small longitudinal disturbances such that each of them proceeds to perform a harmonic vibration; prove that there will be two waves of vibrations, in opposite directions, with the same velocity $a(\lambda/mc)^{\frac{1}{2}}\,n/\pi\sin\pi/n$, where $a$ is the average distance between two successive particles, $n$ the number of intervals between two particles in the same phase, and $\lambda$ the modulus of elasticity.

4216. (The late M. COLLINS, LL.D.)—A straight plank, $a$ feet in length, has one end on a horizontal plane, and the other against a vertical wall, its angle of inclination to the vertical wall being $\beta$. A ball begins to move from rest from the upper end of the plank, and at the same instant the lower end begins to slide out from the wall with a uniform velocity. Required the curve the ball describes in space, and the inclination of the plank to the wall when the ball arrives at the end of it.

4217. (J. J. WALKER, M.A., F.R.S.)—If three conics have a point in common, this will be a double point on their Jacobian. Prove that, conversely, if the Jacobian have a double point, either the three conics meet in this point, or the condition $\mathrm{T}^2=16\mathrm{M}$ (according to Dr. SALMON's notation for the invariants of the system) must hold.

4219. (N'IMPORTE.)—Given the radii of three circles inscribed in a triangle, so that each circle touches the other two, and two sides of the triangle; to find the sides of the triangle.

4220. (A. RENSHAW.)—Let ABC be a triangle and DE a tangent to its inscribed circle, and also parallel to the side BC. Then, if EZ be drawn parallel to AB and meeting BC in Z, and ZD be joined and produced to meet AC in W, and BW be joined cutting ED produced in X, prove that EX = AQ, Q being the point in which the inscribed circle touches AC. Also, prove other properties of the lines in the diagram. (See the Solution of Question 4036.)

4221. (N'IMPORTE.)—To construct the triangle there are given the radii of the three circles inscribed in the three triangles formed by drawing lines parallel to each side of the triangle, meeting the other two sides, and touching the circumference of the inscribed circle. (See the Solution of Question 4036.)